# Giving in Networks *

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**Abstract**

This paper uses a network approach to study the giving behavior of self-interested individuals motivated by social relations. Our theory accommodates the well-defined productive networks that characterize modern economies, differentiates the production network from the social context in which agents interact, and treats the production network as a different object from the giving network. We show that voluntary giving can arise among selfish agents who do not maintain any direct pre-existing productive relationship. We also provide conditions under which some agents never receive voluntary gifts from other members of the society. The model also illustrates how the social context endogenously determines who are the givers and the receivers.

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# 1 Introduction

In the *Theory of Moral Sentiments* Adam Smith asks :"Why do people give away wealth for the good of others?". This question appears to deeply contrast with Smith's egoistic and market-oriented representation of individual behavior in *The Wealth of Nations*. More than two hundred years after Adam Smith's seminal works, the study of the relation between self-interested individuals and voluntary giving remains relevant for academics and policy makers. This long lasting research has differentiated altruistic giving (Kolm, 1966) from non-altruistic giving. The motives for the latter type of giving are non-altruistic normative, self-interest, and social effects.[1] In this paper we build on the self-interested aspect of transfers to study how underlying social structures affect giving. Thus, we focus on the social effect motive for giving caused by social relations.[2] We use networks to model the social relations of self-interested individuals, some of whom receive a negative shock on their welfare. The framework provided by our theory allows us to study how the combination between the underlying social structure and a shock on the welfare of some of its members, determine the pattern of the voluntary redistribution of welfare through giving.[3] This giving pattern forms a network of transfers or *giving network*.

Our paper's main contribution is to provide a general but tractable framework to understand the role of the interaction between a production network, the welfare it generates, and a shock on the welfare of selfish agents on their voluntary transfers. The latter triplete is referred as the social context. This contribution stems from the fact that our theory contains three distinguishing elements. The first element is the accommodation of the well-defined productive networks that characterize modern economies in a general theory of giving. In modern economies, agents –people, firms, or countries– interact in a variety of production networks, where they carry out market and non-market exchanges and collect an output from peer-to-peer interactions. For instance, firms exchange goods and services in complex networks; countries are interconnected by financial and trade networks; and individuals maintain productive links

[1] See Kolm (2006) for a broader discussion of non-altruistic motives for giving. For an in depth description of altruistic motives for giving see Laferrère and Wolff (2006).

[2] Kolm (2006) argues that the social effect motive for giving based on social relations aims to maintain or initiate a relation.

[3] We focus on voluntary giving as opposed to compulsory giving in the form of taxes. Wicksteed (1910), Pareto (1916), Nash (1950), Kolm (1966), Samuelson (1954), and Becker (1974) study the relation between voluntary and compulsory giving. For a broad discussion on the latter topic see Ythier (2006).

with a subset of coworkers at the workplace. Family and friendship ties also form complex networks and individuals collect non-market goods such as love, support, or advice from those interactions. In general, almost any type of human action can indeed be thought in terms of such production networks.

The second distinguishing element of our theory is that it differentiates the production network from the social context in which agents interact. Consider the following example. Suppose a production network with agents (individuals, firms, or countries) $A$, $B$, and $C$ and a given amount of resources owned by each of them. Suppose a case where agent $A$ is severely hit by a tragic event and, thus, $B$ and $C$ become potential givers of transfers. Now suppose a second case where $B$ and $C$ is more severely harmed than $A$, which converts them in the potential receivers of transfers from $A$. The comparison between these two cases illustrates that the roles of receivers and givers emerge from the interaction between a production network and a shock. Therefore, given a production network the same agent may assume the role of a giver or the role of a receiver depending on the shock she suffers. Thus, who are the givers and the receivers in a network is not confined to the production network alone but to the whole social context.

The third distinguishing element of our model is that it treats the production network and the giving network as two different objects. The definition of gift is compatible with observing direct transfers from agents indirectly related or even not related in the production network. Consider again the above example. Even though $A$ could have no productive links with $C$, a transfer could flow from $A$ to $C$. In other words, a perfect overlap between the production network and the giving network is not necessary, and the latter network may include the transfer of both market and non-market goods.[4]

The conjuction of these three characteristic elements of our theory accomodates (i) the exchange of market and non-markets goods in productive networks, (ii) the separation between the production network and the social context, and (iii) the possibility of giving among agents that do not maintain a direct pre-existing productive relationship.

We use our model to show how the topology of a network interacts with a shock to

[4]Remittances and inheritances are examples of monetary gifts. Humanitarian programs, a free teaching lesson, or an invitation for dinner are possible examples of non-monetary gifts. An advice or providing healing to someone are examples of non-market gifts.

induce the role of a voluntary giver as opposed to agents having fixed roles independent from the social context in which the agents act. In our model, agents derive revenues from their interaction with other agents in a given initial production network. The production network is hit by heterogeneous exogenous shocks. The shocks, the production network, and the revenues generated by the production network, induce two classes of agents: poor and rich. If a shock is large enough, then it destroys productive links. Well-off agents can sustain some of her productive links by forming a giving network through which they directly transfer resources to a subset of the poor agents. Therefore, rich and poor agents are determined by the location and intensity of the shock entering the network. Rich agents play the role of givers and poor agents receive the transfers from the rich.

Using this network-based approach to giving, first we show how giving can arise from the interaction of selfish agents that only aim to maximize the revenues they collect in a given production network. The latter occurs because agents seek to maintain direct and indirect productive social relations. Thus, giving can reach agents located at the maximal distance in the production network from the giver. Our model also implies that the location and intensity of a shock's entrance to the production network and the production network itself determine the giving network. This occurs because our model endogenously determines which agents are givers and the potential receivers. Moreover, we find general conditions under which some agents never receive transfers from any giver. In addition, we provide general conditions under which all the links in the giving network exist in the production network. Analogously, we find general conditions under which some link in the giving network does not exist in the production network. Our model also has important implications for the empirical analysis of giving because, as we show later, the observation of a giving network does not identify the underlying production network. Lastly, we prove that in complex production networks focalized transfers sustain the whole network.

Finally, this paper contributes to a wide range of applications where social relations motivate gifts that take the form of direct monetary, non-monetary, or non-market transfers. Some of the areas of these applications are corporate ownwership and control (Dixit, 1983; Fama and Jensen, 1983), financial stability (Acemoglu et al., 2015), and family economics (Becker, 1976 and 1981). This wide scope of areas of contribution stems from three properties of giving: it does not necessary imply reciprocity, giving can be carried out between agents who do not hold any direct pre-existing productive relationship, and giving can involve the transfer of market and non-market goods.

The remainder of the paper is structured as follows. Section 2 presents the model and the characterization of an individual's optimal behavior in the model. Section 3 delves into some of giving patterns implied by the model. In Section 4 the main results of the paper are explained. Section 5 discusses different applications of the theory to family economics, corporate governance, empirical analysis of giving, and financial rescues. Finally, Section 6 concludes.

# 2 The Model

In this section we build the model that we use to study the social motives for giving by focusing on the underlying social context and its implied giving network. Going forward, first we define several network theory concepts that we use throughout the paper. Second, we define a social structure as the formal expression of the social context. Third, we introduce the building block of the model: the definition of a layer in a social structure. These two definitions imply a two-classes society with rich and poor agents, where the former choose how much to give to the latter. Next, we explain how agents' gifts affect the social structure by changing the layers. Then, we define a giving agent's payoff. Finally, we analyze a giving agents' optimization problem, which generates the giving agent's direct transfers or giving decision. The solution to this problem describes the giving network in a single–rich–agent social structure, or the best response when there are multiple rich agents.

## 2.1 Preliminary definitions

A set of nodes $N$ contains elements indexed $1, 2, 3, ..., n$, where $n$ denotes the cardinality of $N$. A dyadic relation, or link, between two different nodes $i$ and $j$ in $N$ is denoted by $ij$. The set of links between two nodes in $N$ is $G$. Thus, a network $g$ is a pair $(N, G)$. The existence of the link $ij$ in $g$ is denoted as $ij \in g$. The network $g$ is undirected if $ij = ji$.[5] The set of all

[5]We adopt the convention that $ii \notin g$. In addition, a directed network is such that $ij \neq ji$.

possible networks on $N$ is $\mathcal{G}(N)$. The network where there are no links between any two nodes in $N$ is called the empty network and it is denoted by $g^{\varnothing} = (N, G^{\varnothing})$, where $G^{\varnothing} = \varnothing$. Node $i$'s neighborhood in $g$ is $\eta_i(g) = \{j \in N : ij \in G\}$. If $ij \in g$, then $i$ and $j$ are involved in $ij$. The set of links in which the nodes in $I \subseteq N$ are involved is $L(I) = \{ij \in G : j \in \bigcup_{i \in I} \eta_i(g)\}$. The subnetwork of the nodes that belong to $N_s \subseteq N$ in $g$ is $g(N_s) = \Big(N_s, G - L\big(N - N_s\big)\Big)$ and the $N_s$-subnetwork of nodes that belong to $N_s \subseteq N$ in $g$ is $g[N_s] = \Big(N, G - L\big(N - N_s\big)\Big)$.[6] We alternatively denote a subnetwork $g'$ of $g$ as $g' = (N^{g'}, G^{g'})$, where $N^{g'} \subseteq N$ and $G^{g'} \subseteq G$.

A path in a network is a finite sequence of links that connect nodes that do not repeat.[7] The set of paths that connect an initial node $i$ and terminal node $i'$ in $g$ is $\Theta_{ii'}(g) = \{g(N_s) \in \mathcal{G}(N_s) : g(N_s)$ is a path between $i$ and $i'$ such that $i, i' \in N_s\}$. If $i = i'$, then $\Theta_{ii}(g^0) = \{(\{i\}, \varnothing)\}$. That is, we assume there is no link from an agent to herself. To ease notation, we define a path between the initial node $i$ and the terminal node $i'$ in network $g$ as $\theta_{ii'} = \theta$ such that $\theta \in \Theta_{ii'}(g)$. We define the distance between $i$ and $i'$ as the geodesic distance between $i$ and $i'$. That is,

$$d_{ii'}(g) = \begin{cases} \min\limits_{\theta \in \Theta_{ii'}(g)} \#G^{\theta} & \text{if } \Theta_{ii'}(g) \neq \varnothing \text{ and } i \neq i' \\ \infty & \text{if } \Theta_{ii'}(g) = \varnothing \text{ and } i \neq i' \\ 0 & \text{if } i = i'. \end{cases}$$

Agents $i$ and $i'$ are disconnected in $g$ if, and only if $d_{ii'}(g) = \infty$, and they are directly connected in $g$ if, and only if $d_{ii'}(g) = 1$.[8] Finally, the addition of two networks $g$ and $g'$ is $g + g' = \big(N^g \cup N^{g'}, G^g \cup G^{g'}\big)$.

## 2.2 A social structure

We assume there is a set of agents $N = \{1, ..., n\}$ and an ex ante undirected production network $g^0 = (N, G^0)$.[9] Link $ij \in g^0$ generates welfare to agents $i$ and $j$ in the form of revenues. The revenues produced by link $ij \in g^0$ to agent $i$ from agent $j$ is $y_i^j > 0$. If $ij \notin g^0$, then $y_i^j = y_j^i = 0$. Each agent $l \in N$ has an ex ante revenue-endowment $y_l^l \geq 0$, henceforth the endowment. The

[6] A $N_s$-subnetwork is also called partial network in Berge (2001).

[7] According to Jackson (2008): "A path may also be defined to be a subnetwork that consists of the set of involved nodes and the set of links between these nodes."

[8] Agents $i$ and $i'$ are connected in $g$ if, and only if $\Theta_{ii'}(g) \neq \varnothing$.

[9] We assume that all the networks in the paper are undirected.

revenue matrix $Y \in \mathbb{R}_+^{n^2}$ describes the revenue sources of each agent in the ex ante production network. That is, $ij \in g^0$ implies that element $ij$ in $Y$ is $y_i^j$, $ij \notin g^0$ implies that element $ij$ in $Y$ is zero, and element $ii$ in $Y$ is agent $i$'s endowment, $y_i^i$. For a fixed ex ante production network $g^0$ and its corresponding revenue matrix $Y$, $\Pi_l\left(g^0, Y\right) = P_l(g^0, Y) - \underline{y}$ is agent $l$'s ex ante payoff under $g^0$, where $P_l : \mathcal{G}(N) \times \mathbb{R}_+^{n^2} \to \mathbb{R}_+$ such that $P_l\left(g^0, Y\right) = y_l^l + \sum_{l' \in \eta_l(g^0)} y_l^{l'}$ is $l$'s ex ante total revenue, and $\underline{y} \geq 0$ is $l$'s subsistence level (which is homogeneous across agents). That is, agent $l$'s total revenue under $g^0$ is exclusively derived from her ex ante endowment and $l$'s direct interactions with her neighbors. Therefore, the elimination of a link in $g^0$ reduces the revenues for at least two agents. Assumption 1 formalizes the idea that ex ante and for each agent, $g^0$ generates a total revenue that is at least as large as the agents' subsistence level.

**Assumption 1.** $\Pi_l\left(g^0, Y\right) \geq 0$ *for all* $l \in N$.

The ex ante production network receives an exogenous shock $\epsilon = (\epsilon_1, ..., \epsilon_n)$, which simultaneously affect all the agents. Agent $l$'s shock on her ex ante payoff is $\epsilon_l \in \mathbb{R}$. Thus, agent $l$'s interim payoff under $g^0$ is $\Pi_l\left(g^0, Y\right) - \epsilon_l$. If $\Pi_l\left(g^0, Y\right) - \epsilon_l < 0$, then agent $l$'s revenues under $g^0$ cannot meet the subsistence level $\underline{y}$. In this case, we say that $l$ dies. The death of an agent has two consequences. First, each of $l$'s links are eliminated, which implies that for $l$ and each of $l$'s neighbors the revenues generated by the former links are lost. Second, $l$'s endowment, $y_l^l$, is destroyed, which implies that none of the surviving agents can use the endowment of a dead agent, i.e. endowments are non-transferable after death. We assume, however, that an agent's endowment is transferable while still alive.

**Definition 1.** *A social structure is a triplete* $\alpha = \left(g^0, Y, \epsilon\right)$ *such that* $\alpha \in \mathcal{G}(N) \times \mathbb{R}_+^{n^2} \times \mathbb{R}^n$.

Therefore, a social structure is composed by the ex ante production network $g^0$, the revenues derived from the interactions of agents in $g^0$ denoted by $Y$, and the shock vector $\epsilon$. Hence, a social structure is the formal expression of the social context. Absent any giving, if for some agent $l$ her interim payoff is such that $\Pi_l\left(g^0, Y\right) - \epsilon_l < 0$, then $g^0$ cannot be the ex post production network. In the next section we study how transfers affect the agents' interim payoffs, thereby affecting the ex post production network.

## 2.3 Social structure's layers

In this section we show that there exists a causal order in which agents in the ex ante production network die due to the shock. This causal order shows how directly or indirectly a shock reaches an agent that dies. We define the first layer of agents that die in $\alpha = (g^0, Y, \epsilon)$ as the set $\mathcal{S}^1(\alpha) = \{l \in N : \Pi_l(g^0, Y) - \epsilon_l < 0)\}$. That is, $\mathcal{S}^1(\alpha)$ is the set of agents that die as a direct consequence of the shock $\epsilon$ in $\alpha$. Because each $l \in \mathcal{S}^1(\alpha)$ dies, by definition, the set of links of all the agents in $\mathcal{S}^1(\alpha)$ are eliminated from $g^0$, generating an interim $N_s$-subnetwork of $g^0$ denoted by $g^1(\alpha) = \Big(N, G^0 - L\big(\mathcal{S}^1(\alpha)\big)\Big)$. That is, $g^1(\alpha)$ is a $\mathcal{S}_1^c$-subnetwork of $g^0$, i.e. $g^1(\alpha) = g_0[\mathcal{S}_1^c(\alpha)]$, which implies that each agent $l$ in $g^1(\alpha)$ obtains an interim payoff of $\Pi_l(g^1(\alpha), Y) - \epsilon_l$.[10] Thus, the second layer of agents that die after the shock vector $\epsilon$ reaches $g^0$ is $\mathcal{S}^2(\alpha) = \{l \in N : \Pi_l(g^1(\alpha), Y) - \epsilon_l < 0\}$. The agents in $\mathcal{S}^2(\alpha)$ do not die directly due to the shock, but they die as a consequence of the death of the agents in the layer that precedes $\mathcal{S}^2(\alpha)$, i.e $\mathcal{S}^1(\alpha)$. Analogously, the $q$'th layer of agents that die due to $\epsilon$ in $g^0$ is $\mathcal{S}^q(\alpha) = \{l \in N : \Pi_l(g^{q-1}(\alpha), Y) - \epsilon_l < 0\}$, where $g^{q-1}(\alpha) = \Big(N, G^0 - L\big(\bigcup_{m \in \{1,\cdots,q-1\}} \mathcal{S}^m(\alpha)\big)\Big)$.

Therefore, the sequence of layers describes the interdependency between the survival of different sets of agents in a social structure $\alpha$. Agents in the first layer are those who die as direct consequence of the shock. Agents in the second layer are those who cannot survive in the shocked ex ante production network without their interactions with the agents in the first layer. An analogous interpretation applies to the subsequent layers of agents who die. The set of all the agents that die in $\alpha$ is $\mathcal{P}(\alpha) = \bigcup_{\mathcal{S}^l(\alpha) \neq \varnothing} \mathcal{S}^l(\alpha)$. We refer to $\mathcal{P}(\alpha)$ as the set of poor agents in the social structure $\alpha$. If $l \in N$ is not poor, then $l$ is rich. The set of rich agents is $\mathcal{K}(\alpha)$ and if there are $K_\alpha$ rich agents, then there are $n - K_\alpha$ poor agents. Therefore, for fixed $g^0$ and $Y$, different $\epsilon$ define different rich and poor agents sets. Hence, the latter sets are an outcome of the social structure, as opposed to exogenous sets. We define that rich agents are givers and poor agents are receivers.

Lastly, a social structure's topology is a triplete that completely describes the consequences of a shock on the ex ante underlying production network, holding the revenues $(Y)$ fixed. That is, the social structure's topology of $\alpha \in \mathcal{G}(N) \times \mathbb{R}_+^{n^2} \times \mathbb{R}^n$ is $\omega(\alpha) = (\{S^i(\alpha)\}_{S^i \neq \varnothing}, \mathcal{K}(\alpha), G^0)$. Therefore, $\omega(\alpha)$ characterizes the causal order in which poor agents die due to the shock to

[10] We define $\mathcal{S}_1^c(\alpha) = N - \mathcal{S}^1(\alpha)$.

$g^0$ under $Y$, and who are the rich agents in the ex ante production network who survive and have the choice of giving in $\alpha$.[11] The set $\Omega(N) = \{\omega(\alpha) : \alpha \in \mathcal{G}(N) \times \mathbb{R}_+^{n^2} \times \mathbb{R}^n\}$ is the set of all possible social structures' topologies on $N$. Hereafter, we focus the analysis in social structures where there exists at least one rich agent and one poor agent, which is defined by the set $\mathcal{A} = \{\alpha \in \mathcal{G}(N) \times \mathbb{R}_+^{n^2} \times \mathbb{R}^n : \mathcal{K}(\alpha) \neq \varnothing \text{ and } \mathcal{P}(\alpha) \neq \varnothing\}$.

## 2.4 Social structure's layers with transfers

Now we study the effect of transfers or gifts on the social structure's layers.[12] The direct transfer $t_k^p$ is the gift that rich agent $k \in \mathcal{K}(\alpha)$ gives to poor agent $p \in \mathcal{P}(\alpha)$ in the social structure $\alpha \in \mathcal{A}$. Thus, $t_k^p$ is the $p$th component of the transfer vector $t_k \in \mathbb{R}_+^{n-K_\alpha}$. The $\mathcal{K}(\alpha)$–subnetwork or network of rich agents is $g^{\mathcal{K}(\alpha)} = \big(N, G^0 - L\big(\mathcal{P}(\alpha)\big)\big)$. Then, agent $k$'s feasible transfer set is $T_k(\alpha) = \left\{t_k \in \mathbb{R}_+^{n-K_\alpha} : \sum_{p\in\mathcal{P}(\alpha)} t_k^p \leq \Pi_k\left(g^{\mathcal{K}(\alpha)}, Y\right) - \epsilon_k\right\}$.[13] The set of transfer profiles in $\alpha$ is $\tau(\alpha) = (t_{k=1}, t_{k=2}, \cdots, t_{k=K_\alpha})$. The aggregate transfer vector is $t = \sum_{k\in\mathcal{K}(\alpha)} t_k$. We denote by $t^p$ the $p$th component of vector $t \in \mathbb{R}_+^{n-K_\alpha}$, which contains the total transfers made by all the rich agents to poor agent $p \in \mathcal{P}(\alpha)$.

The rich agents' aggregate transfers have the potential of saving poor agents from death. Therefore, rich agents are capable of affecting the ex post production network and, thereby, their own revenues. The first layer of poor agents that die under the aggregate transfer vector $t$ in $\alpha$ is $\mathcal{S}_t^1(\alpha) = \left\{l \in N : \Pi_l(g^0, Y) - \epsilon_l + \sum_{k\in\mathcal{K}(\alpha)} t_k^l < 0)\right\}$. Therefore, all the links of agents in $\mathcal{S}_t^1(\alpha)$ are eliminated from $g^0$ generating the interim network $g_t^1(\alpha) = \Big(N, G^0 - L\big(\big(\mathcal{S}_t^1(\alpha)\big)\big)\Big)$. Then the set $\mathcal{S}_t^2(\alpha) = \left\{l \in N : \Pi_l\left(g_t^1(\alpha), Y\right) - \epsilon_l + \sum_{k\in\mathcal{K}(\alpha)} t_k^l < 0\right\}$ is the second layer of poor agents that die with transfers $t$ in $\alpha$. Analogously, the $q$'th layer of poor agents that die with transfers $t$ in $\alpha$ is $\mathcal{S}_t^q(\alpha) = \left\{l \in N : \Pi_l\left(g_t^{q-1}(\alpha), Y\right) - \epsilon_l + \sum_{k\in\mathcal{K}(\alpha)} t_k^l < 0\right\}$, where $g_t^{q-1}(\alpha) = \Big(N, G^0 - L\big(\bigcup_{m\in\{1,\cdots,q-1\}} \mathcal{S}_t^m(\alpha)\big)\Big)$. Thus, the social structure's layers with transfers $t$ may not coincide with social structure's layers absent any transfer described in the previous section. Next, we describe how the rich agents' possibility of affecting $\alpha$'s layers under transfers

[11] Notice that the definition of a social structure's topology implies that two different social structures could exhibit the same topology.

[12] Throughout the paper we use the term "transfer" and the term "gift" equivalently.

[13] An alternative definition of $k$'s feasible transfer set to be considered is $T_k(\alpha) = \left\{t_k \in \mathbb{R}_+^{n-K_\alpha} : \sum_{p\in\mathcal{P}(\alpha)} t_k^p \leq \Pi_k(g^{\varnothing}(\alpha), Y) - \epsilon_k\right\}$. The latter definition implies that the endowment of a rich agent and the revenues derived by her from the links with other rich agents are not perfect substitutes for the poor agents.

determines their payoffs.

## 2.5 Agents' payoffs in a social structure with transfers

First, we define the set of all the $N_s$-subnetworks of the ex ante production network, $\mathbb{G}(g^0) = \{g^0[N_s] \in \mathcal{G}(N): N_s \subseteq N\}$. The function $H : \{g^0\} \times \mathbb{R}^n \times \mathbb{R}_+^{n^2} \times \mathbb{R}_+^{n-K} \to \mathbb{G}(g^0)$ such that $H(t,\alpha) = g^0\Big[\Big(\bigcup_{\mathcal{S}_t^l(\alpha)\neq\emptyset} \mathcal{S}_t^l(\alpha)\Big)^c\Big]$ is the ex post production network with transfers $t$, which describes the effect of $t$ on $g^0$ in $\alpha$. Therefore, agent $l$'s ex post payoff in $\alpha$ is a function of $t$ such that

$$\pi_l(t,\alpha) = \begin{cases} \Pi_l\left(H(t,\alpha), Y\right) - \epsilon_l + \sum\limits_{k\in\mathcal{K}(\alpha)} t_k^l & \text{if } l \in \mathcal{P}(\alpha) \text{ and } l \in \Big(\bigcup\limits_{\mathcal{S}_t^l(\alpha)\neq\emptyset} \mathcal{S}_t^l(\alpha)\Big)^c \\ \Pi_l\left(H(t,\alpha), Y\right) - \epsilon_l - \sum\limits_{p\in\mathcal{P}(\alpha)} t_l^p & \text{if } l \in \mathcal{K}(\alpha) \\ -\underline{y} - \epsilon_l & \text{if } l \in \mathcal{P}(\alpha) \text{ and } l \in \bigcup\limits_{\mathcal{S}_t^l(\alpha)\neq\emptyset} \mathcal{S}_t^l(\alpha). \end{cases}$$

The latter function captures both the effect of the aggregate transfer vector $t$ on the ex ante production network and the fact that the death of an agent results in the complete loss of her revenues from all sources.

## 2.6 The rich agent's giving decision

In this section, we set up the problem of a rich agent to accommodate the analysis of social structures with one or more rich agents. We illustrate the latter case by focusing the analysis on the pure-strategy Nash equilibria of the direct simultaneous transfer game the rich agents play in $\alpha$, which we denote in normal form $\Gamma(\alpha, \mathcal{K}(\alpha), \bigtimes_{k\in\mathcal{K}(\alpha)} T_k(\alpha), \{\pi_k\}_{k=1}^{K_\alpha})$. Let $t_{-k} = t - t_k$ and $\phi(t_k, t_{-k}) = t_k + t_{-k}$.

**Definition 2.** *For a fixed* $\alpha \in \mathcal{A}$, $\tau^* \in \bigtimes_{k\in\mathcal{K}(\alpha)} T_k(\alpha)$ *is an equilibrium transfer profile of* $\Gamma(\alpha, \mathcal{K}(\alpha), \bigtimes_{k\in\mathcal{K}(\alpha)} T_k(\alpha), \{\pi_k\}_{k=1}^{K_\alpha})$ *if* $\pi_k\left(\phi(t_k^*, t_{-k}^*), \alpha\right) \geq \pi_k\left(\phi(t_k, t_{-k}^*), \alpha\right)$ *for all* $t_k \in T_k(\alpha)$ *and all* $k \in \mathcal{K}(\alpha)$.

The outcome of $\Gamma$ is the giving network. The latter is implied by the optimal giving decision of the single rich agent when $K_\alpha = 1$ or it is implied by the equilibrium transfer profile

$\tau^* \in \bigtimes_{k \in \mathcal{K}(\alpha)} T_k(\alpha)$ when $K_\alpha > 1$.[14]

Now we study the giving decision of a rich agent. Fix $\alpha \in \mathcal{A}$, $k \in \mathcal{K}(\alpha)$, and $t_{-k} \in \mathbb{R}_+^{n-K_\alpha}$. Then, the problem of a rich agent $k$ is

$$\max_{t_k \in T_k(\alpha)} \pi_k\big(\phi(t_k, t_{-k}), \alpha\big) = \Pi_k\Big(H\big(\phi(t_k, t_{-k}), \alpha\big), Y\Big) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k^p. \tag{1}$$

The continuity of $\pi_k$ on $t_k$ and the compactness of $T_k(\alpha)$ imply that a solution to problem (1) exists. We solve problem (1) in an economically meaningful and systematic two step procedure. First, we identify an efficient transfer vector to sustain a network. Then, we find a profit maximizing network conditional on transfer efficiency.

To identify a cost-efficient transfer vector, we define two instrumental sets. These sets are the set of all the $N_s$-subnetworks of $g^0$ that are sustainable by some non-negative $t$ in $\alpha$, denoted by $\mathcal{X}(\alpha)$, and the set of $N_s$-subnetworks that are sustainable by a resource-unconstrained rich agent $k \in \mathcal{K}(\alpha)$ for a given $t_{-k}$, denoted by $\mathcal{X}(\alpha, t_{-k})$.[15] The set $\mathcal{X}(\alpha)$ fixes the set of achievable networks in the social structure. Fixing $t_{-k}$, an analogous interpretation holds for $\mathcal{X}(\alpha, t_{-k})$. Therefore, for fix $t_{-k}$ and $\overline{g} \in \mathcal{X}(\alpha, t_{-k})$, each element of $k$'s set of efficient transfer vectors that sustains $\overline{g}$ solves

$$\begin{aligned} \max_{t_k \in \mathbb{R}_+^{n-K_\alpha}} \pi_k\big(\phi(t_k, t_{-k}), \alpha\big) = \Pi_k\Big(H\big(\phi(t_k, t_{-k}), \alpha\big), Y\Big) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k^p \\ s.t. \\ H\big(\phi(t_k, t_{-k}), \alpha\big) = \overline{g}. \end{aligned} \tag{2}$$

By the definition of $H$, for all $\overline{g} \in \mathcal{X}(\alpha, t_{-k})$ there exists $\bar{t}_k \in \mathbb{R}_+^{n-K_\alpha}$ such that $H\big(\phi(t_k, t_{-k}), \alpha\big) = \overline{g}$ implies $t_k^p \in [\bar{t}_k^p, \infty)$ for each $p \in \mathcal{P}(\alpha)$. The latter implies that for every $\overline{g} \in \mathcal{X}(\alpha, t_{-k})$ a solution to problem (2) exists, because $\pi_k$ is linear and strictly decreasing in each $t_k^p \in [\bar{t}_k^p, \infty)$.

Let $\tilde{t}_k(\alpha, \overline{g}, t_{-k})$ be a solution to (2). Then, $\tilde{t}_k(\alpha, \overline{g}, t_{-k})$ must minimize $k$'s total transfers to sustain $\overline{g}$. That is, $\sum_{p \in \mathcal{P}(\alpha)} \tilde{t}_k^p(\alpha, \overline{g}, t_{-k}) \leq \sum_{p \in \mathcal{P}(\alpha)} t_k'^{\,p}$ for all $t_k' \in \mathbb{R}_+^{n-K_\alpha}$ such that

[14] When there is a single rich agent, we let $t_{-k} \in \mathbb{R}_+^{n-1}$ be such that $t_{-k} = (0, \dots, 0) = \vec{\mathbf{0}}$.

[15] Technically, $\mathcal{X}(\alpha) = \{g \in \mathbb{G}(g^0) : g = H(t, \alpha) \text{ and } t \in \mathbb{R}_+^{n-K_\alpha}\}$, and $\mathcal{X}(\alpha, t_{-k}) = \{g \in \mathbb{G}(g^0) : g = H\big(\phi(t_k, t_{-k}), \alpha\big) \text{ and } t_k \in \mathbb{R}_+^{n-K_\alpha}\}$ for $t_{-k} \in \mathbb{R}_+^{n-K_\alpha}$. For $t_{-k}$ such that $t_{-k}^p$ is sufficiently large for each $p \in \mathcal{P}(\alpha)$ implies that $g^0 \notin \mathcal{X}(\alpha, t_{-k})$ and $g^0 \in \mathcal{X}(\alpha)$.

$\overline{g} = H\big(\phi(t'_k, t_{-k}), \alpha\big)$. Therefore, a rich agent's transfers to a poor agent that are greater than the amount of resources needed by the latter to stay alive are not efficient. This inefficiency occurs because a lower amount can accomplish the same objective. The latter also implies that the solution to problem (2) can be characterized in terms of the poor agents' subsistence needs, as we next show.

We define poor agent $p$'s subsistence needs in an arbitrary production network $g$ when she receives transfers $\overline{t} \in \mathbb{R}_+$, under a fix revenue matrix $Y \in \mathbb{R}_+^{n^2}$ and a vector shock $\epsilon \in \mathbb{R}_+^n$ as $r_p(\alpha, g, \overline{t}) = \max\big\{\epsilon_p - \Pi_p\big(g, Y\big) - \overline{t}, 0\big\}$. That is, $r_p(\alpha, g, \overline{t})$ are the resources that $p$ needs to survive in $g$ when she receives transfers $\overline{t}$. We use the definition of $r_p$ to characterize an efficient transfer vector to sustain $\overline{g}$ in problem (2).

**Lemma 1.** *Fix $\alpha \in \mathcal{A}$, $k \in \mathcal{K}(\alpha)$, $t_{-k} \in \mathbb{R}_+^{n-K_\alpha}$, and $\overline{g} \in \mathcal{X}(\alpha, t_{-k})$. Suppose $\tilde{t}_k(\alpha, \overline{g}, t_{-k})$ solves problem (2) for $k$. Then, for all $p \in \mathcal{P}(\alpha)$,*

$$\tilde{t}_k^p\left(\alpha, \overline{g}, t_{-k}\right) = \begin{cases} 0 & \textit{if } \eta_p\left(\overline{g}\right) = \varnothing \\ r_p(\alpha, \overline{g}, t_{-k}^p) & \textit{if } \eta_p\left(\overline{g}\right) \neq \varnothing. \end{cases}$$

The intuition of Lemma 1 is as follows. Suppose that $\tilde{t}_k$ solves (2). Now, assume that $p$ has some neighbor in $\overline{g}$. In our model, the existence of each link in any production network solely depends on the subsistence of the two agents that are linked. Therefore, the definition of $\pi_l$ implies that for fixed transfers for each poor agent other than $p$, a transfer $t_k^p$ that is strictly larger than $p$'s subsistence needs, is not optimal. This occurs because for any fixed production network $\overline{g}$, any transfer of the latter class sustains exactly the same ex post production network as $\tilde{t}_k$ at a larger cost to $k$. Thus, it generates a payoff to $k$ that is strictly lower than just transferring $p$'s subsistence needs of resources to $p$. However, if $p$ has no neighbor in $\overline{g}$, then it is not optimal for $k$ to keep $p$ alive. Therefore, a positive transfer from $k$ to $p$ implies a strictly lower payoff to $k$ than $k$ not transferring any resources to $p$ at all.

We use Lemma 1 to define $k$'s set of feasible networks for fixed $t_{-k}$, $\mathcal{X}_f(\alpha, t_{-k})$.[16] Next, we find $k$'s optimal sustainable and feasible production network by considering only the efficient

[16]Technically, $\mathcal{X}_f(\alpha, t_{-k}) = \{g \in \mathcal{X}(\alpha, t_{-k}) : \tilde{t}_k(\alpha, g, t_{-k}) \in T_k(\alpha)\}$ for $t_{-k} \in \mathbb{R}_+^{n-K_\alpha}$.

and feasible transfer vector associated to $g \in \mathcal{X}_f(\alpha, t_{-k})$. That is, we solve

$$\max_{g \in \mathcal{X}_f(\alpha, t_{-k})} \tilde{\pi}_k\Big(\alpha, g, t_{-k}\Big) = \Pi_k\Big(g, Y\Big) - \epsilon_k - \textstyle\sum_{p \in \mathcal{P}(\alpha)} \tilde{t}_k^p(\alpha, g, t_{-k}). \tag{3}$$

A solution to problem (3) exists because the set $\mathcal{X}_f(\alpha, t_{-k})$ is finite and there exists $\tilde{\pi}_k \in \mathbb{R}$ for each $g \in \mathcal{X}_f(\alpha, t_{-k})$. Therefore, Lemma 1 and the solution to problem (3) directly characterizes the solution to a rich agent's problem.

**Proposition 1.** *For fixed $\alpha \in \mathcal{A}$, $k \in \mathcal{K}(\alpha)$, and $t_{-k} \in \mathbb{R}_+^{n-K_\alpha}$, $\tilde{t}_k(\alpha, g^*, t_{-k})$ solves problem (2) and $g^*$ solves problem (3) if, and only if, $t_k^* = \tilde{t}_k(\alpha, g^*, t_{-k})$ solves problem (1) .*

Proposition 1 states that the solution set of problem (1) is characterized by the properties of the solutions to problems (2) and (3). Thus, rich agent $k$'s transfer choice can be understood as solving the complementary subproblems (2) and (3). Therefore, for a fixed social structure and other rich agents' transfers, $k$'s best response is to make efficient and feasible transfers to sustain the production network that gives her the highest payoff. Therefore, an equilibrium transfer profile $\tau^*$ is such that $t_k^*(\alpha, t_{-k}^*) = \tilde{t}_k(\alpha, g^*, t_{-k}^*)$ and $\tilde{\pi}_k(\alpha, g^*, t_{-k}^*) \geq \tilde{\pi}_k(\alpha, g, t_{-k}^*)$ for all $g \in \mathcal{X}_f(\alpha, t_{-k}^*)$ and all $k \in \mathcal{K}(\alpha)$.

Finally, the undirected network formed by the rich agents' transferring decisions to poor agents in a social structure $\alpha$ is a giving network

$$g_T = \begin{cases} \Big(N, \{ij\}_{t_i^j > 0}\Big) & \text{if } t_i^j > 0 \text{ for some pair } (i, j) \in N^2 \\ (N, \varnothing) & \text{otherwise.} \end{cases}$$

That is, a non–empty giving network $g_T$ is a pair that specifies the $n$ agents that populate $\alpha$ and a set of undirected links between some agents in the set of givers
$E(g_T) = \{i \in N : t_i^j > 0 \text{ for some } j \in N\}$ and some agents in the set of the receivers
$U(g_T) = \{i \in N : t_j^i > 0 \text{ for some } j \in N\}$. The set of the equilibrium giving networks of a social structure $\alpha$ is a correspondance $\Phi : \mathcal{A} \to \mathcal{G}(N)$.[17]

[17]The existence of a solution to problem (1) implies that if $K_\alpha = 1$, then $\Phi$ is non-empty. If $K_\alpha > 1$, we assume that the existence of a solution to problem (1) implies that $\Phi$ is non-empty. Later in the paper, we provide examples that illustrate the existence of pure strategy Nash equilibria.

# 3 Giving Behaviors

In this section, our goal is to convey the inherent complexity of the causal relation between pre-existing social structures and their implied giving behavior. We do so by illustrating giving in specific social structures. First, we present a single-giver social structure topology example. Then, we discuss strategic interactions within social structures with multiple givers.

Let us start by considering a social structure $\alpha'$ with a topology defined as $\omega(\alpha') = \left(\left\{S^1(\alpha') = \{z\}, S^2(\alpha') = \{j,i\}\right\}, \mathcal{K}(\alpha') = \{k\}, G^0 = \{kj, iz, jz\}\right)$. In $\omega(\alpha')$, the ex ante production network is $g^0 = (\{k,j,i,z\},\{kj,iz,jz\})$. The effect of the shock vector on $g^0$ given $Y$ directly causes the death of agent $z$, thus eliminating the productive links $iz$ and $jz$, and generating the interim network $g^1 = (\{k,j,i,z\},\{kj\})$. That is, agent $z$ is in the first layer of $\alpha'$. Under $g^1$ agents $j$ and $i$ die. Hence, $j$ and $i$ are in $\alpha'$s second layer. The implications of the effects of the shock on $g^0$ stop when the remaining link $(kj)$ disappears, thereby generating the empty network $g^{\varnothing}$. It follows that agents $z$, $j$, and $i$ are poor agents whereas $k$ is the single rich agent in $\alpha'$.

In this single-giver social structure $t_{-k} = (0,...,0) = \vec{\mathbf{0}}$[18] and, thus, $\mathcal{X}(\alpha') = \mathcal{X}(\alpha', \vec{\mathbf{0}})$. The set of all the $N_s$-subnetworks of $g^0$ that can be sustained by non-negative transfers in $\alpha'$ is $\mathcal{X}(\alpha') = \{g^0, g^1, g^{\varnothing}\}$. We use now Lemma 1 to characterize the cost-effective transfer vector, $\tilde{t}_k(\alpha', g, \vec{\mathbf{0}})$, for each $g \in \mathcal{X}(\alpha')$. The latter that is the solution to problem (2).

We start by analyzing the cost-effective transfer vector to sustain the ex ante production network. Agent $k$'s gifts are contained in $t_k = \left(t_k^z, t_k^j, t_k^i\right)$. Lemma 1 directly implies that $\tilde{t}_k(\alpha', g^0, \vec{\mathbf{0}}) = (r_z(\alpha', g^0, 0), r_j(\alpha', g^0, 0), r_i(\alpha', g^0, 0))$ with $r_z(\alpha', g^0, 0) = \epsilon_z + \underline{y} - y_z^z - y_z^j - y_z^i$ and $r_j(\alpha', g^0, 0) = r_i(\alpha', g^0, 0) = 0$. Notice that the ex ante production network of $\alpha'$ can be preserved with an exclusive transfer to $z$. Lemma 1 states that the cost-effective gift to $z$ in $\alpha'$ is equal to the $z'$s subsistence needs. Any transfer strictly greater than $z$'s subsistence needs would also keep $z$ alive, but at a higher cost. On the other hand, a transfer smaller than $z$'s subsistence needs causes $z$ to die. Therefore, the cost-effective way for $k$ to sustain $g^0$ is by making transfers to $z$ such that $z$'s subsistence needs are exactly covered. In addition, $z$ is the single agent located in the first layer. Therefore, if $z$ lives, all the other poor agents in $\alpha'$ also stay alive. Hence,

[18] See footnote 13.

the resource needs for $j$ and $i$ under $g^0$ are null. Thus, the cost-effective gifts for these agents involve zero transfers.

Let us now focus on $k$'s cost-efficient form to sustain $g^1$. Lemma 1 implies that the cost-effective transfer vector to sustain $g^1$ considers null transfers to $z$ and $i$ and transfers that match $j$'s subsistence needs under $g^1$. The $N_s$-subnetwork $g^1$ does not contain the productive links that involve either $z$ or $i$. Thus, positive transfers to $z$ or $i$ would not be a cost-effective way for $k$ to sustain $g^1$. The rich agent, however, transfers a positive amount to $j$, which equal $j$'s subsistence needs. The intuition of the latter is analogous to the one discussed in the previous paragraph. Therefore, $\tilde{t}_k(\alpha', g^1, \vec{\mathbf{0}}) = (0, r_j(\alpha', g^0, 0), 0)$ with $r_j(\alpha', g^1, 0) = \epsilon_j + \underline{y} - y_j^j - y_j^k$. Lastly, the rich agent could choose $g^{\varnothing}$. In this case, Lemma 1 implies that $\tilde{t}_k(\alpha', g^{\varnothing}, \vec{\mathbf{0}}) = (0,0,0)$ by an analogous argument as in the previous two cases.

Having solved problem (2), we define the set of $N_s$-subnetworks that contains only networks that are sustainable and feasible for $k$ in $\alpha'$. That is, $\mathcal{X}_f(\alpha', \vec{\mathbf{0}}) = \{g^0, g^1, g^{\varnothing}\}$.[19] Then, $k$ must choose a network in the set $\mathcal{X}_f(\alpha', \vec{\mathbf{0}})$. The latter choice is the rich agent's solution to problem (3). When choosing an $N_s$-subnetwork under $\alpha'$, the rich agent considers that she has a unique productive link in the ex ante production network: $kj$. Thus, when choosing between $g^0$, $g^1$, and $g^{\varnothing}$, the rich agent's tradeoff considers the benefits of $kj$, i.e. $y_k^j$ or 0, and the cost of sustaining $kj$, i.e. either the cost of sustaining $g^0$ or $g^1$.

Suppose the rich agent decides to sustain her productive link with $j$. She can do so by transferring resources to $z$ or $j$. Agent $k$ has no direct pre-existing relation with $z$ in $g^0$. On the other hand, agent $j$ does have a direct pre-existing relation to $k$ in the ex ante production network. Agent $k$'s minimum cost of a life-saving transfer to $z$ is $\epsilon_z + \underline{y} - y_z^z - y_z^j - y_z^i$, and $k$'s cost efficient life-saving transfer to $j$ is $\epsilon_j + \underline{y} - y_j^j - y_j^k$. These cost-effective transfer vectors imply that either $g^0$ or $g^1$ are sustained. However, $k$'s revenue from her productive link with $j$, $y_k^j$, could be small compared with the cost of keeping alive either $j$ or $z$–at the minimum cost–. In this case, $k$ does not become a giver in $\alpha'$ and the cost of this action for her is simply zero.

[19]Equivalently, $\mathcal{X}_f(\alpha', \vec{\mathbf{0}}) = \{g \in \mathcal{X}(\alpha') : \sum_{p \in \{j,i,z\}} \tilde{t}_k^p(\alpha', g, \vec{\mathbf{0}}) \leq y_k^k - \underline{y} - \epsilon_k\}$.

Using Proposition 1, we characterize the equilibrium giving network of $\alpha'$:

$$g_T(\alpha') = \begin{cases} (\{k,j,i,z\},\{kz\}) & \text{if } \; r_z(\alpha',g^0,0) \leq \min\left\{r_j(\alpha',g^1,0), y_k^j, y_k^k - \underline{y} - \epsilon_k\right\} \\ (\{k,j,i,z\},\{kj\}) & \text{if } \; r_j(\alpha',g^1,0) \leq \min\left\{r_z(\alpha',g^0,0), y_k^j, y_k^k - \underline{y} - \epsilon_k\right\} \\ (\{k,j,i,z\},\{\varnothing\}) & \text{else.} \end{cases} \tag{4}$$

Equation (4) allows the illustration of important aspects of the relation between a pre-existing social structure and their implied giving network. Suppose that $\alpha'$ is such that is optimal for $k$ to save agent $z$:

(a) $r_z(\alpha',g^0,0) < r_j(\alpha',g^1,0)$ (it is less expensive to save $z$ than $j$),

(b) $r_z(\alpha',g^0,0) < y_k^j$ (the revenues for $k$ from her link with $j$ are greater than the cost of keeping alive $z$), and

(c) $r_z(\alpha',g^0,0) < y_k^k - \underline{y} - \epsilon_k$ (feasibility).

Conditions (a) through (c) imply that an equilibrium giving network of $\alpha'$ is $g_T(\alpha') = (\{k,j,i,z\},\{kz\})$. There are several economic properties of $g_T(\alpha')$ that are worth further discussing.

First, the equilibrium giving network is not an $N_s$-subnetwork of the ex ante production network. Namely, the equilibrium giving network connects agents who do not have any direct pre-existing relation in the production network. This undocking between the giving network and the ex ante production network limits the external validity of studies on giving that exclusively occurs between agents with a direct pre-existing social relation. For expositional purposes, suppose that a researcher tries to extract information on the giving behavior of agent $k$ by exclusively studying her gifts to $j$, with whom $k$ is directly connected in the production network. The researcher would conclude that $k$ is not a giver since she does not observe any transfer from $k$ to $j$. However, this conclusion is invalid since $k$ is a giver, only that her transfers go to $z$ an not to $j$. Therefore, this example suggests that placing the analysis of the social effects motives for giving in the social context is crucial.

Second, under $\alpha'$ we have that conditions (a) through (c) imply that the giving network sustains the entire ex ante production network. Moreover, the causal order in which the poor agents die in $\alpha'$ implies that the preservation of the first layer sustains all the remaining layers. Therefore, every agent in $\alpha'$ survives, which implies that the entire ex ante production network is sustained. This occurs because $z$ is the single agent located in the first layer. Hence, the current example suggests that gifts sustaining the first layer, as opposed to transfers to all the poor agents, are sufficient to sustain the entire ex ante production network.

Third, the existence of a causal order in which agents die in a given social structure, unveils sufficient conditions on the social structure's topology for the existence of segregation in private giving. In this example, the mere existence of two, or more, layers of poor agents is a sufficient condition to make some poor agents not to receive positive transfers from the rich agent. Suppose that all the poor agents are sustained by the giving of $k$. Then, all the agents located in the first layer are kept alive and the entire ex ante production network would be preserved. Therefore, it can not be optimal for the rich agent to make transfers to all the poor agents in $\alpha'$.

Let us now focus on agent $i$. In $\alpha'$, there is no path in $g^0$ such that $i$ is located between the rich agent and the first layer of poor agents. Hence, $i$ is completely irrelevant for $k$ to sustain her link with $j$. Therefore, transferring resources to $i$ is never optimal for $k$.[20]

Finally, two different social structures can generate the same equilibrium giving network. Let social structure $\alpha'' \neq \alpha'$ be such that $g_T(\alpha'') = g_T(\alpha')$. Suppose $\omega(\alpha'') = \left(\left\{S^1(\alpha'') = \{z, j, i\}\right\}, \mathcal{K}(\alpha'') = \{k\}, G^0 = \{kz\}\right)$. Following analogous steps to those previously described in the analysis of $\alpha'$, we can characterize the solution to the problem of the rich agent in $\alpha''$ as

$$g_T(\alpha'') = \begin{cases} (\{k, j, i, z\}, \{kz\}) & \text{if } \; r_z(\alpha, g^0, 0) \leq \min\left\{y_k^z, y_k^k - \underline{y} - \epsilon_k\right\} \\ (\{k, j, i, z\}, \{\varnothing\}) & \text{else.} \end{cases} \tag{5}$$

If $r_z(\alpha'', g^0, 0) < \min\left\{y_k^z, y_k^k - \underline{y} - \epsilon_k\right\}$, expression 5 indicates that $g_T(\alpha'') = (\{k, j, i, z\}, \{kz\})$, which implies that $g_T(\alpha'') = g_T(\alpha')$. The economic relevance of the latter observation is that two different social structures are observationally equivalent with respect to the equilibrium giving network that they form.

[20]This fact does not implies that agent $i$ dies.

*Strategic Interactions*

In this subsection, we illustrate how to use our theory to study strategic interactions between rich agents. We discuss some new insights that stem from this analysis. We conclude by showing that the results extracted from the analysis of $\alpha'$ are also present in some social structures which exhibit strategic interactions.

Consider a social structure $\tilde{\alpha}$ characterized by $\omega(\tilde{\alpha}) = \left(\left\{S^1(\tilde{\alpha}) = \{j\}\right\}, \mathcal{K}(\tilde{\alpha}) = \{k_1, k_2\}, G^0 = \{k_1 j, k_2 j\}\right)$. Thus, there are two rich agents in $\tilde{\alpha}$ ($k_1$ and $k_2$) and each of them is connected to the same poor agent ($j$). The direct transfer game in which $k_1$ and $k_2$ participate is $\Gamma_{k_1,k_2} = \Gamma(\tilde{\alpha}, \{k_1, k_2\}, T_{k_1}(\tilde{\alpha}) \times T_{k_2}(\tilde{\alpha}), \{\pi_{k_1}(t, \tilde{\alpha}), \pi_{k_2}(t, \tilde{\alpha})\})$. We focus the analysis on the pure-strategy Nash equilibria of $\Gamma_{k_1,k_2}$. Thus, $k_1$ and $k_2$ must non-cooperatively and simultaneously decide the transfers to $j$. Notice that, if aggregate transfers ($t^j$) are equal or greater than $j$'s subsistence needs ($r_j(\tilde{\alpha}, g^0, 0) = \underline{y} + \epsilon_j - y_j^j - y_j^{k_1} - y_j^{k_2}$), then $t^j$ sustains the ex ante production network. If the latter does not occur, the empty network $g^{\varnothing}$ is generated. Therefore, the set of $N_s$-subnetworks that can be sustained in $\tilde{\alpha}$ by a non-negative aggregate transfer vector contains the ex ante production network and the empty network. That is, $\mathcal{X}(\tilde{\alpha}) = \{g^0, g^{\varnothing}\}$.

The problems solved by $k_1$ and $k_2$ are symmetric. In addition, for both rich agents the strategy space is a strict subset of $\mathbb{R}_+$, i.e. $t_k$ and $t_{-k}$ are non-negative scalars. Following the procedure described in Section 2.6, we analyze how agent $k \in \mathcal{K}(\tilde{\alpha})$ solves problem (2). Agent $k$ must choose the cost-effective gifts to sustain each of the $N_s$-subnetworks in $\mathcal{X}(\tilde{\alpha}, t_{-k}) \subseteq \mathcal{X}(\tilde{\alpha})$. The cost-effective way to preserve $g^0$ for $k$ is through gifts that exactly cover $j$'s subsistence needs, net of the resources transferred by $-k$.[21] Therefore, $\tilde{t}_k(\tilde{\alpha}, g^0, t_{-k}) = r_j(\tilde{\alpha}, g^0, t_{-k})$, where $r_j(\tilde{\alpha}, g^0, t_{-k}) = r_j(\tilde{\alpha}, g^0, 0) - t_{-k}$. In addition, it is straightforward to conclude that the cost-effective transfer for $k$ to sustain the empty network is zero, for any amount of the other rich agent's transfer. That is, $\tilde{t}_k(\tilde{\alpha}, g^{\varnothing}, t_{-k}) = 0$ for any $t_{-k} \in \mathbb{R}_+$.

Once that each rich agent's cost-effective way of sustaining each $N_s$-subnetwork in $\mathcal{X}(\tilde{\alpha}, t_{-k})$ is computed, the feasible set $\mathcal{X}_f(\tilde{\alpha}, t_{-k})$ is determined.[22] Then, for each rich agent, problem (3) is solved following analogous steps as for the single-giver case. As stated by Proposition 1, this procedure computes rich agent $k$'s best response transfers to rich agent $-k$'s transfers.

[21] See Lemma 1 and its interpretation.

[22] In this case, $\mathcal{X}_f(\tilde{\alpha}, t_{-k}) = \{g \in \mathcal{X}(\tilde{\alpha}, t_k) : \tilde{t}_k(\tilde{\alpha}, g^0, t_{-k}) \leq y_k^k - \underline{y} - \epsilon_k\}$.

In order to characterize the latter solution, we define a technical threshold that represents an upper bound for the optimality of $k$'s transfers. Agent $k$'s optimal transfers cannot exceed neither $y_k^j$ nor $y_k^k - \underline{y} - \epsilon_k$. A transfer greater than $y_k^j$ is not optimal because the amount of the gift would exceed the revenues collected from the productive link that it sustains. In addition, a gift greater than $y_k^k - \underline{y} - \epsilon_k$ would exceed the amount of resources that $k$ has available to transfer. Let $\bar{t}_k(\tilde{\alpha})$ be the maximum amount that makes $k$'s transfer to $j$ optimal. Then, $\bar{t}_k(\tilde{\alpha}) = \min\left\{y_k^j, y_k^k - \underline{y} - \epsilon_k\right\}$ for $k \in \{k_1, k_2\}$. Thus, in $\tilde{\alpha}$, the optimal transfers of $k$ to $-k$'s transfers are

$$t_k^{br}(\tilde{\alpha}, t_{-k}) = \begin{cases} r_j(\tilde{\alpha}, g^0, t_{-k}) & \text{if } \; r_j(\tilde{\alpha}, g^0, t_{-k}) \leq \bar{t}_k(\tilde{\alpha}) \\ 0 & \text{else.} \end{cases} \tag{6}$$

In this case, by construction, $t_k^{br}(\tilde{\alpha}, t_{-k})$ is a function for $k \in \{k_1, k_2\}$. We use now the latter best response function characterization to provide the solution to the transfer game $\Gamma_{k_1,k_2}$.

First, suppose that $\bar{t}_{k_1} + \bar{t}_{k_2} < r_j(\tilde{\alpha}, g^0, 0)$.[23] Then, the sum of the maximum amount of resources that is optimal to transfer by the rich agents is not enough to sustain $j$ in the ex ante production network. Therefore, the rich agents are not wealthy enough to keep $j$ alive, or the productive link with $j$ is not valuable enough as to encourage them to sustain the poor agent in ex ante production network. Therefore, there is no giving in this case.

Suppose now that $\bar{t}_{k_1} + \bar{t}_{k_2} \geq r_j(\tilde{\alpha}, g^0, 0)$, $\bar{t}_{k_1} < r_j(\tilde{\alpha}, g^0, 0)$ and $\bar{t}_{k_2} < r_j(\tilde{\alpha}, g^0, 0)$. In this case, the sum of the maximum amount of resources that is optimal for the rich agents to transfer to $j$ is enough to keep the poor agent alive. However, it is not individually optimal to sustain $j$. Here, the strategic interaction between the rich agents triggers two types of equilibria. The first type of equilibrium is characterized by having each rich agent transferring a strictly positive amount of resources to $j$. Therefore, each reach agent has a link with $j$ in the giving network. In the second equilibrium, the agents transfer no resources to $j$, which implies that the equilibrium giving network is the empty network.

A third case is $\bar{t}_{k_1} \geq r_j(\tilde{\alpha}, g^0, 0)$ and $\bar{t}_{k_2} \geq r_j(\tilde{\alpha}, g^0, 0)$. These conditions imply that, even though $t_{-k} = 0$, it is optimal for $k$ to sustain $j$. Therefore, giving is always observed in this type of social structures. Moreover, multiple equilibria also exist. Thus, this third case highlights that it is possible to observe a dissimilar giving behavior among identical agents.

[23] Here, we use $\bar{t}_{k_i} = \bar{t}_{k_i}(\tilde{\alpha})$ to ease notation.

Lastly and without loss of generality, suppose that $\bar{t}_{k_1} + \bar{t}_{k_2} \geq r_j(\tilde{\alpha}, g^0, 0)$, $\bar{t}_{k_1} \geq r_j(\tilde{\alpha}, g^0, 0)$, and $\bar{t}_{k_2} < r_j(\tilde{\alpha}, g^0, 0)$. In this case, it is optimal for $k_1$ to make transfers to $j$ even though $k_2$ makes no gifts. Therefore, $k_1$ can individually sustain $j$, hence forming a link with the poor agent in the giving network, which is not possible for $k_2$. The condition $\bar{t}_{k_1} > \bar{t}_{k_2}$ implies that agent $k_1$ can optimally afford bigger gifts to $j$ than $k_2$. We interpret the latter as $k_1$ having a comparative advantage in giving with respect to $k_2$. Moreover, we can observe specialization in the production of giving when $k_1$ individually sustain $j$ in the ex ante production network. Notice the specialized agent is precisely who carries comparative advantages in the production of giving ($k_1$). The formal analysis for the previous four paragraphs is in Appendix A.

We now provide an example to show that the results derived for $\alpha'$ can also be present in social structures that exhibit strategic interactions. Consider a social structure $\hat{\alpha}$ with a topology $\omega(\hat{\alpha}) = \left(\left\{\mathcal{S}^1(\hat{\alpha}) = \{z\}, \mathcal{S}^2(\hat{\alpha}) = \{j, i\}\right\}, \mathcal{K}(\hat{\alpha}) = \{k_1, k_2\}, G^0 = \{k_1 j, k_2 j, iz, jz\}\right)$. Suppose that

(i) $y^{k_1}_{k_1} + y^{k_2}_{k_2} - 2\underline{y} - \epsilon_{k_1} - \epsilon_{k_2} < \underline{y} + \epsilon_j - y^j_j - y^{k_1}_j - y^{k_2}_j$ (it is not feasible for the rich agents to sustain $j$),

(ii) $y^{k_1}_{k_1} - \underline{y} - \epsilon_{k_1} > r_z(\hat{\alpha}, g^0, 0)$ (it is feasible for $k_1$ to individually sustain $z$),

(iii) $y^{k_2}_{k_2} - \underline{y} - \epsilon_{k_2} > r_z(\hat{\alpha}, g^0, 0)$ (it is feasible for $k_2$ to individually sustain $z$),

(iv) $y^j_{k_1} > r_z(\hat{\alpha}, g^0, 0)$ (it is optimal for $k_1$ to sustain $z$),

(v) $y^j_{k_2} > r_z(\hat{\alpha}, g^0, 0)$ (it is optimal for $k_2$ to sustain $z$),

Condition (i) implies that aggregate resources are not enough to cover $j$'s subsistence needs. Conditions (ii) and (iii), on the other hand, imply that it is feasible for each rich agent to sustain $z$ in the ex ante production network. Moreover, conditions (iv) and (v) imply that it is also optimal for them to sustain $z$. By sustaining $z$ each of the rich agents can preserve the productive link with $j$ and the value of that link is greater than $z$'s subsistence needs. Moreover, a direct implication of the topology structure of $\hat{\alpha}$ is that gifts to $i$ do not allow the subsistence of $j$ but the subsistence of $z$ ensures that both $j$ and $i$ stay alive. Therefore, agent $i$ is irrelevant for the rich agents to preserve the productive link they have with $j$.

The analysis of this social structure follows the steps described in this subsection. We have that conditions (i) through (v) imply that $\Phi(\tilde{\alpha}) = \{g^1, g^2, g^3\}$ with $g^1 = (\{k_1, k_2, j, i, z\}, \{k_1 z\})$, $g^2 = (\{k_1, k_2, j, i, z\}, \{k_2 z\})$, and $g^3 = (\{k_1, k_2, j, i, z\}, \{k_1 z, k_2 z\})$. That is, the equilibrium giving network correspondence includes giving networks in which either one rich agent makes transfers to $z$ or both do so.

Notice that the equilibrium giving network does not resemble the ex ante production network in $\hat{\alpha}$. That is, even though $z$ does not have any type of direct pre-existing relation with neither $k_1$ nor $k_2$ in the ex ante production network, one or both of them optimally make a gift to $z$. Moreover, private giving sustains the entire ex ante production network because the subsistence of $z$ is sufficient for the survival of $j$ and $i$ given the structure of layers that characterize $\hat{\alpha}$. In addition, some poor agents in $\hat{\alpha}$ are segregated from giving. The intuition of the latter result is exactly the same as the one developed above for the case of a single-giver social structure: sustaining the first layer of agents is sufficient to sustain the entire network and, thus, no agent rich agent will find optimal to transfer resources to all the poor individuals. The social structure $\hat{\alpha}$ also shows that there is a second reason why a poor agent could be segregated from giving, which concerns to agent $i$: the subsistence of this agent is irrelevant for sustaining $k_1$'s and/or $k_2$'s link with $j$. All these results were already derived for a single-giver social structure and we have shown now that they can also be observed in a multi-givers social structure.

We end the discussion of this section by illustrating how two different multi-givers social structures can produce exactly the same equilibrium giving network. Consider a social structure $\hat{\alpha}'$ with topology $\omega(\hat{\alpha}') = \left(\left\{\mathcal{S}^1(\hat{\alpha}') = \{z, j, i\}\right\}, \mathcal{K}(\hat{\alpha}') = \{k_1, k_2\}, G^0 = \{k_1 z, k_2 z\}\right)$. Suppose

(I) $y_{k_1}^{k_1} - \underline{y} - \epsilon_{k_1} > r_z(\tilde{\alpha}, g^0, 0)$ (it is feasible for $k_1$ to individually sustain $z$),

(II) $y_{k_2}^{k_2} - \underline{y} - \epsilon_{k_2} > r_z(\tilde{\alpha}, g^0, 0)$ (it is feasible for $k_2$ to individually sustain $z$),

(III) $y_{k_1}^{z} > r_z(\tilde{\alpha}, g^0, 0)$ (it is optimal for $k_1$ to sustain $z$),

(IV) $y_{k_2}^{z} > r_z(\tilde{\alpha}, g^0, 0)$ (it is optimal for $k_2$ to sustain $z$),

Then, under conditions (I) to (IV), we have that $\Phi(\hat{\alpha}') = \Phi(\hat{\alpha})$ even though the geometry of ex ante production network in these social structures is different. Therefore, the result regarding the fact that two different social structures are observationally equivalent with respect to the equilibrium giving network that they induce can also be derived for a multi-givers social structure.

# 4 Results

In this section we generalize the insights that stemmed from the analysis carried out in Section 3. We start by defining some concepts that we will use to state and analyze the main results of the paper.

We define the *direct diffusion network* in the social structure $\alpha$ as $DIF(\alpha) = \sum_{k \in \mathcal{K}(\alpha)} \sum_{j \in \mathcal{S}^1(\alpha)} \sum_{\theta \in \Theta_{kj}(g^0)} \theta$. The set of agents who do not belong to $N^{DIF(\alpha)}$, or *ramified agents*, is $N^{R(\alpha)} = \mathcal{P}(\alpha) - N^{DIF(\alpha)}$.[24] The set of agents that connect the direct diffusion network to its ramified agents in $g$ given $\alpha$, or *frontier*, is $\tilde{\mathcal{P}}(\alpha, g) = \{p \in \mathcal{P}(\alpha) \cap N^{DIF(\alpha)} : \eta_p(g) \cap N^{R(\alpha)} \neq \varnothing\}$. An element of $\tilde{\mathcal{P}}(\alpha, g)$ is a *frontier agent* in $g$ given $\alpha$. The set of ramified agents that stems from a frontier agent $p'$ in $g$ given $\alpha$ is $\hat{\mathcal{P}}(\alpha, g, p') = \{p \in N^{R(\alpha)} : \Theta_{pp'}(g) \neq \varnothing \text{ for } p' \in \tilde{\mathcal{P}}(\alpha, g) \text{ and } d_{pp'}(g) \leq d_{pp''}(g)\ \forall\ p'' \in \tilde{\mathcal{P}}(\alpha, g)\}$. A *ramification* of $p' \in \tilde{\mathcal{P}}(\alpha, g^0)$ is a subnetwork $g^0\left(p' \cup \hat{\mathcal{P}}(\alpha, g^0, p')\right)$.

We use Lemmas 3 through **??** in Appendix B to prove Proposition 2 ahead. Altogether, these technical lemmas are used to show that the survival of a frontier agent keeps all the agents in its ramification alive. In addition, the survival of ramified agents who are disconnected from the direct diffusion network does not affect the rich agents' payoffs. Hence, it is not optimal for the rich agents to make gifts to ramified agents.

**Proposition 2.** *For a fixed social structure $\alpha$, a poor agent receives strictly positive transfers only if she is in the direct diffusion network.*

Proposition 2 characterizes *where* in the ex ante production network gifts are received. Concretely, it states that transfers are allocated to agents in the direct diffusion network. This *location-based* characterization of the receivers, directly implies that ramified agents are excluded/segregated from the rich agents' gifts. However, being segregated from the rich agents' giving is compatible with some ramified agents' survival as long as the corresponding agents in

[24]Recall that according to the definition provided in section 2.2, $DIF(\alpha) = (N^{DIF(\alpha)}, G^{DIF(\alpha)})$.

the frontier survive.

However, Proposition 2 does not imply that the existence of ramified agents is the sole sufficient condition for rich agents' segregated giving behavior. Even in social structures with no ramified agents, some poor agents in the direct diffusion network could be segregated from receiving strictly positive transfers from some rich agent in equilibrium.

**Proposition 3.** *A multilayer social structure topology implies that there is at least one poor agent that does not receive positive transfers in equilibrium.*

Positive gifts that keep alive all the poor agents of a social structure imply, by construction, the survival of all the agents located in the first layer. In the latter case, the productive links between the agents in the first layer and those located in the successive layers are sustained. Thus, all the poor agents that are not located in the first layer stay alive even without receiving gifts from the rich agents. Therefore, an equilibrium giving network cannot exhibit positive transfers to all the poor agents of a multilayer social structure.

So far, we have analyzed how a social structure causes giving. However, one could also ask what can be learned about the social structure from an observed equilibrium giving network. Each of the following three propositions (4, 5, and 6) study the extent of the informational content of an observed (equilibrium) giving network regarding the ex ante production network. We discuss how these propositions bring implications for the empirical analysis of giving in Section 5.

The set of all the social structures where all the poor agents are either disconnected or directly connected with the rich agents in the ex ante production network is $\tilde{\mathcal{A}}$.[25] The ex ante production networks of the social structures in $\tilde{\mathcal{A}}$ correspond to the types of relations studied by Becker (1976, 1981) in the context of altruism. The following proposition states the information that can be extracted from a giving network induced by social structures in $\tilde{\mathcal{A}}$.

**Proposition 4.** *In social structures where all the poor agents are either disconnected or directly connected with the rich agents in the ex ante production network, the equilibrium giving network is such that there are no links that do not exist in the ex ante production network.*

Proposition 4 states that, for each social structure in $\tilde{\mathcal{A}}$, the set of links of the equilibrium

[25] That is, $\tilde{\mathcal{A}} = \{\alpha \in \mathcal{A} : d_{pk}(g^0) = \{1, \infty\}$ for all $p \in \mathcal{P}(\alpha)$ and all $k \in \mathcal{K}(\alpha)\}$.

giving network is a subset of the links of the ex ante production network. This occurs because the survival of poor agents that are disconnected from the rich agents does not affect the payoffs of the latter individuals. Thus, it is not optimal for any rich agent to transfer resources to agents with whom there is not a direct or indirect relation. This implies that only poor agents that are directly connected to rich agents receive gifts in social structures in $\tilde{\mathcal{A}}$. This result contains two economic implications. First, transfers between two agents informs on the existence of a productive link between these individuals in the ex ante production network. Second, observing transfers from a rich agent to a directly connected poor agent in $g^0$ characterizes the giving behavior of the rich agent in the entire social structure: there is no giving beyond a rich agent's neighborhood in the ex ante production network.

However, the social structures in $\tilde{\mathcal{A}}$ are not adequate for describing complex social structures.[26] This fact raises the question about the limitations of the informational content of giving behavior with respect to the ex ante production network in less constrained social structures than those in $\tilde{\mathcal{A}}$.

**Proposition 5.** *There exists some social structure such that some of its equilibrium giving network contains a link between agents that are not directly connected in the ex ante production network.*

Proposition 5 implies that an observed transfer from one rich agent to a poor agent does not provide certainty about the existence of a link between these agents in the ex ante production network. In addition, the giving behavior cannot be characterized by observing transfers between neighboring agents in $g^0$. The latter is consequence on the fact that giving can occur beyond close relations, as our examples of Section 3 already illustrated. Therefore, Proposition 5 warns about potential biases when empirically studying giving solely in the context of direct relations.

Propositions 4 and 5 highlighted that the equilibrium giving network is not sufficient to infer neither the social structure nor the ex ante production network. In addition, there is another motive for caution when inferring pre-existing relations from an equilibrium giving network.

[26] The set of complex social structures is $\tilde{\mathcal{A}}^c$.

**Proposition 6.** *For each social structure in $\mathcal{A}$ with three or more agents, there exists a social structure with a different underlying ex ante production network such that both induce the same equilibrium giving network.*

Thus, Proposition 6 shows that even though the observed giving network could convey information on the ex ante production network, it will never be enough to completely infer $g^0$. It follows that that different social structures are observationally equivalent regarding the giving network they induce: any observed transfers can be induced by two different ex ante production networks.[27]

Finally, we show that strictly positive transfers to all the poor agents that populate a social structure is not necessarily a cost-effective way of sustaining the entire production network.

**Proposition 7.** *In any social structure, transfers to all the non isolated poor agents located in the first layer, equal or greater than their subsistence needs, are sufficient to sustain the entire ex ante production network.*

Proposition 7 is a direct consequence of the structure of layers intrinsic to any social structure. The layers in a social structure determine the casual order in which agents in the ex ante production network die due to the shock. Concretely, they determine the group of agents that die as direct consequence of the shock that hits the economy and the group of agents that die as consequence of the disappearance of the productive links that they have with the former agents. Then, positive transfers that sustain a subset of agents can be sufficient to prevent the death of individuals who do not die as a direct consequence of the shock.

# 5 Discussion

The examples we develop ahead highlight how the theory presented in this paper has the potential to enrich the study of several economics phenomena. First, we sketch an application of our theory to study how non-altruistic motives affect giving in the context of the family. Then, we suggest how the context of a firm may affect the firm's decisions on corporate ownership and control. Third, we discuss how our framework could be applied for the analysis of optimal

[27] It is trivial to obtain analogous results to Proposition 6 by marginally changing $Y$ or $\epsilon$.

rescue-policies in complex financial networks. Finally, we highlight the kind of biases that ignoring the social context of giving decisions introduce in experiments that study giving.

*Family Economics*

Becker provided the first formal analysis of giving within a family.[28] The motive for giving in Becker's analysis arises from parents' altruistic preferences. One could wonder whether altruistic preferences are needed to observe intra-family transfers. Our theory shows that transfers within the family can arise from social motives. We also highlight that the preservation of family relations may express itself in transfers beyond the family's sphere. To illustrate these insights, consider the social structure $\alpha'$, which was studied in Section 3. Let us interpret the rich agent $k$ as the parent and the poor agent $j$ as the child. These agents are directly connected in an ex ante production network and collect some market or non-market goods from that relation; for instance, love. In our model, transfers from the parent to the child are not motivated by Beckerian altruistic preferences. What motives these transfers is the preservation of the productive link that the parent has with the child. Moreover, our Proposition 4 implies that, to preserve that link, a parent could transfer resources outside the family circle.[29] Our theory constitutes a non-exclusive alternative to Becker's analysis of giving within the family.

*When Does Corporate Ownership Induce Corporate Control?*

The separation between corporate ownership and corporate control is one of the oldest issues discussed in the corporate governance literature.[30] Demsetz and Lehn (1985) made early efforts to study how corporate ownership causes corporate control by describing the market for corporate control. More recently, some efforts have been made to describe de consequences

[28] See, for instance, Becker (1976), Becker (1981), Becker and Tomes (1986), Becker and Barro (1988), among others.

[29] For instance, to agent $z$ in $\alpha'$.

[30] Vitali et al. (2011) define corporate control as "*the chances of seeing one's own interest prevailing in the business strategy of the firm*" whereas simple ownership does not imply such influence in the firm's strategy. Several papers study the differences between ownership and control (Cantillo, 1998; Frank and Mayer, 1997). Traditionally the relation between corporate ownership and control has been studied from the perspective of agency costs (Berle and Means, 1932; Jensen and Meckeling, 1976), considering externalities produced by upstream or downstream firms (Dixit, 1983), considering incomplete contracts (Klein et al., 1978; Grossman and Hart, 1986; Hart and Moore, 1990) or from the perspective of the agency problem caused by dispersed ownership (Fama and Jensen, 1983).

of the structure of corporate ownership and control on financial stability (La Porta et al., 1999), with some of them using network theoretical methodologies (Glattfelder, 2010; Vitali et al., 2011). The theory we develop in this paper complements the latter efforts by providing a general framework to study the causal relation from corporate ownership structure to corporate control.

Suppose that the ex ante production network represents the corporate ownership network where each link implies a profit flow from the "owned" firm to the stock holder firm and a capital flow from the stock holder to the "owned" firm. Analogously, suppose that the giving network represents increases of capital for the survival of the "owned" firms due to a direct or indirect shock. An increase in the capital investment of firm $a$ in firm $b$ may lead firm $a$ to control firm $b$. Then, one can use our theory as a framework to understand changes in the network of corporate control as a response for maintaining the profitability of an ex post parent company with respect to a subsidiary. Applying our theory as described above contribute to this corporate ownership and control literature by shedding light on how the roles of the companies might change depending on the nature of the shock that affects the ownership network. This complements the analysis of Shleifer and Vishny (1986) by providing another channel through which dispersed ownership affects corporate control.

*Which Bank is Optimally Saved in a Financial Crisis?*

After the 2008 global financial crisis, the resilience and stability of banking systems have received much attention (Plosser, 2009; Blume et al., 2011). Early studies suggested that the structure of the interbank claims affects the system's resilience (Allen and Gale, 2000; Freixas et al., 2000). More recently, Acemoglu et al., (2015) study financial contagion holding the financial network fixed. Our model complements the latter efforts by suggesting a *rescue-policy* taking the financial network's structure and its associated contagion pattern as given.

Suppose that the ex ante production network represents the financial network, and suppose that the giving network represents a structured collection of rescue packages to troubled banks. Then, our model facilitates the determination of an optimal rescue-policy. Moreover, by considering the existence and properties of the direct diffusion network and the set of ramified agents (Propositions 2, 3, and 7), our framework provides criteria to handle bank defaults in complex financial networks.

*Study of Social Motives for Giving in Experiments*

Field experiments (Frey and Meier, 2004; Armin, 2007; Meier, 2007; Carpenter et al., 2008; Shang and Croson, 2009; DellaVigna et al., 2012; Zarghamee et al., 2017; among others) are frequently used to empirically study giving. These experiments consist on the observation of transfers from one specific group (the treated individuals) to another under factual and counterfactual scenarios. Proposition 5 warns about potential biases when empirically studying giving using small-scale field experiments. This proposition shows that it is the entire social structure what matters to understand giving motivated by social effects. However, it is unlikely that the design considered in a small-scale field experiment captures the entire social structure. This difficulty casts doubts regarding the external validity of the results derived from this empirical methodology when studying the social motives for giving.

# 6 Conclusions

In this paper we develop a general theory of giving in networks. Our model accommodates different aspects that are intrinsic to human societies. First, the exchange of market and non-market goods in networks; second, the complexity of the social context beyond the production network as a determinant of agents' choices; and the imperfect overlap between the production network and the giving network.

The use of networks to model social relations permits a precise characterization of giving behaviors that are motivated by social motives. We show that voluntary giving can arise from selfish agents who do not even maintain a pre-existing productive relationship with the recipients of the gifts. The theory presented in this paper also emphasizes that the location and intensity of an event that hits the production network—what we called “a shock”—determines which agents are givers and the receivers. Moreover, the position of the givers and receivers determine the number, the quantity , and the actual recipients of the gifts. Also our model permits the recognition of general conditions under which some agents are segregated from giving. Lastly, the paper provides general conditions under which focalized transfers sustain the complete production network.

Finally, our theory can be applied to understand a diversity of phenomena that involve the possibility for agents to carry out voluntary transfers. We discussed examples related to the literature on family economics, corporate governance, macro-finance, and field experiments.

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# Appendix A: Main Results

**Lemma 1.** *Fix $\alpha \in \mathcal{A}$, $k \in \mathcal{K}(\alpha)$, $t_{-k} \in \mathbb{R}_+^{n-K_\alpha}$, and $\overline{g} \in \mathcal{X}(\alpha, t_{-k})$. Suppose $\tilde{t}_k(\alpha, \overline{g}, t_{-k})$ solves problem (2) for $k$. Then, for all $p \in \mathcal{P}(\alpha)$,*

$$\tilde{t}_k^p(\alpha, \overline{g}, t_{-k}) = \begin{cases} 0 & \text{if } \eta_p(\overline{g}) = \varnothing \\ r_p(\alpha, \overline{g}, t_{-k}^p) & \text{if } \eta_p(\overline{g}) \neq \varnothing. \end{cases}$$

*Proof.* Fix $\alpha \in \mathcal{A}$, $k \in \mathcal{K}(\alpha)$, $t_{-k} \in \mathbb{R}_+^{n-K_\alpha}$, and $\overline{g} \in \mathcal{X}(\alpha, t_{-k})$. Suppose $\tilde{t}_k(\alpha, \overline{g}, t_{-k})$ solves problem (2) for $k$. Then, $H(\phi(\tilde{t}_k(\alpha, \overline{g}, t_{-k}), t_{-k}), \alpha) = \overline{g}$. For fix $g' \in \mathcal{X}(\alpha, t_{-k})$, the function $\Pi_k(g', Y) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k^p$ and the function $\Pi_p(g', Y) - \epsilon_p + t_{-k}^p + t_k^p \geq 0$ are additively separable in $t_k^p$. Now, pick any $p \in \mathcal{P}(\alpha)$ and let $t_k'(t_k^p) \in \mathbb{R}_+^{n-K_\alpha}$ be such that $t_k'^{\,l} = \tilde{t}_k^l(\alpha, \overline{g}, t_{-k})$ for all $l \neq p$ and $t_k'^{\,p} = t_k^p$ such that $t_k^p \in \mathbb{R}_+$. First, we show that $\eta_p(\overline{g}) = \varnothing$ implies $\tilde{t}_k^p(\alpha, \overline{g}, t_{-k}) = 0$. Then, we show that $\eta_p(\overline{g}) \neq \varnothing$ implies $\tilde{t}_k^p(\alpha, \overline{g}, t_{-k}) = r_p(\alpha, \overline{g}, t_{-k}^p)$.

Suppose that $\eta_p(\overline{g}) = \varnothing$. By definition, $r_p(\alpha, \overline{g}, t_{-k}^p) \geq 0$. Assume $r_p(\alpha, \overline{g}, t_{-k}^p) = 0$. Then, the definition of $r_p$ implies that $\Pi_p(\overline{g}, Y) - \epsilon_p + t_{-k}^p + t_k^p \geq 0$ for all $t_k^p \in [0, \infty)$. Therefore, the definition of $H$ and the construction of $t_k'$ imply that $H(\phi(t_k'(t_k^p), t_{-k}), \alpha) = \overline{g}$ for all $t_k^p \in [0, \infty)$. Hence, by the definition of $\pi_l$, $\pi_k(\phi(t_k'(t_k^p), t_{-k}), \alpha)$ is linear and strictly decreasing for $t_k^p \in [0, \infty)$, which implies that $\tilde{t}_k^p(\alpha, \overline{g}, t_{-k}) = 0$. Now assume that $r_p(\alpha, \overline{g}, t_{-k}^p) > 0$. Then, the definition of $r_p$ implies that $\Pi_p(\overline{g}, Y) - \epsilon_p + t_{-k}^p + t_k^p < 0$ for all $t_k^p \in [0, r_p(\alpha, \overline{g}, t_{-k}^p))$ and that $\Pi_p(\overline{g}, Y) - \epsilon_p + t_{-k}^p + t_k^p \geq 0$ for all $t_k^p \in [r_p(\alpha, \overline{g}, t_{-k}^p), \infty)$. Therefore, the definition of $H$ implies that $H(\phi(t_k'(t_k^p), t_{-k}), \alpha) = \hat{g}$ for all $t_k^p \in [0, r_p(\alpha, \overline{g}, t_{-k}^p))$ and that $H(\phi(t_k'(t_k^p), t_{-k}), \alpha) = \tilde{g}$ for all $t_k^p \in [r_p(\alpha, \overline{g}, t_{-k}^p), \infty)$. Hence, by the definition of $\pi_l$, $\pi_k(\phi(t_k'(t_k^p), t_{-k}), \alpha)$ is linear and strictly decreasing for $t_k^p \in [0, r_p(\alpha, \overline{g}, t_{-k}^p))$ and for $t_k^p \in [r_p(\alpha, \overline{g}, t_{-k}^p), \infty)$, which implies that $\tilde{t}_k^p(\alpha, \overline{g}, t_{-k}) \in \{0, r_p(\alpha, \overline{g}, t_{-k}^p)\}$. Suppose $t_k^p = r_p(\alpha, \overline{g}, t_{-k}^p)$. If $\eta_p(\tilde{g}) \neq \varnothing$, then $\tilde{g} \neq \overline{g}$. Therefore, $t_k'(t_k^p)$ does not solve problem 2. Suppose $\eta_p(\tilde{g}) = \varnothing$. Then, by construction of $t_k'$, $\tilde{g} = \hat{g} = \overline{g}$, which implies that $\pi_k(\phi(t_k'(0), t_{-k}), \alpha) > \pi_k(\phi(t_k'(r_p(\alpha, \overline{g}, t_{-k}^p)), t_{-k}), \alpha)$. Therefore, $t_k'(0)$ is the unique solution to problem (2).

Finally, suppose that $\eta_p(\overline{g}) \neq \varnothing$. By definition $r_p(\alpha, \overline{g}, t_{-k}^p) \geq 0$. Suppose $r_p(\alpha, \overline{g}, t_{-k}^p) = 0$. By construction $\tilde{t}_k^p(\alpha, \overline{g}, t_{-k}) \geq 0$. Therefore $\tilde{t}_k^p(\alpha, \overline{g}, t_{-k}) \geq r_p(\alpha, \overline{g}, t_{-k}^p)$. Now, suppose $r_p(\alpha, \overline{g}, t_{-k}^p) > 0$. By the definition of $r_p$, $t_k^p < r_p(\alpha, \overline{g}, t_{-k}^p)$ implies $\Pi_p(\overline{g}, Y) - \epsilon_p + t_{-k}^p + t_k^p < 0$, which contradicts $\eta_p(\overline{g}) \neq \varnothing$. Therefore, $\tilde{t}_k^p(\alpha, \overline{g}, t_{-k}) \geq r_p(\alpha, \overline{g}, t_{-k}^p)$. Now we complete the proof by showing that $\tilde{t}_k^p(\alpha, \overline{g}, t_{-k}) = r_p(\alpha, \overline{g}, t_{-k}^p)$. The definitions of $\pi_l$ and $r_p$ imply that $\Pi_p(\overline{g}, Y) - \epsilon_p + t_{-k}^p + t_k^p \geq 0$ for all $t_k^p \in \left[r_p(\alpha, \overline{g}, t_{-k}^p), \infty\right)$. Hence, the definition of $H$ and the construction of $t'_k$ imply that $H(\phi(t'_k(t_k^p), t_{-k}), \alpha) = \overline{g}$ for all $t_k^p \in \left[r_p(\alpha, \overline{g}, t_{-k}^p), \infty\right)$. Therefore, the definition of $\pi_l$ implies that $\pi_k(\phi(t'_k(t_k^p), t_{-k}), \alpha)$ is linear and strictly decreasing for $t_k^p \in \left[r_p(\alpha, \overline{g}, t_{-k}^p), \infty\right)$. Thus, by optimality $\tilde{t}_k^p(\alpha, \overline{g}, t_{-k}) = r_p(\alpha, \overline{g}, t_{-k}^p)$. □

**Proposition 1.** *For fixed $\alpha \in \mathcal{A}$, $k \in \mathcal{K}(\alpha)$, and $t_{-k} \in \mathbb{R}_+^{n-K_\alpha}$, $\tilde{t}_k(\alpha, g^*, t_{-k})$ solves problem (2) and $g^*$ solves problem (3) if, and only if, $t_k^* = \tilde{t}_k(\alpha, g^*, t_{-k})$ solves problem (1) .*

*Proof.* Fix $\alpha \in \mathcal{A}$, $k \in \mathcal{K}(\alpha)$, and $t_{-k} \in \mathbb{R}_+^{n-K_\alpha}$. First we prove the only if part. Let $\tilde{t}_k(\alpha, \tilde{g}, t_{-k})$ be a solution to problem (2) for $\overline{g} = \tilde{g}$. Let $t_k^* = \tilde{t}_k(\alpha, g^*, t_{-k})$. Then, $H\big(\phi(t_k^*, t_{-k}), \alpha\big) = g^*$. Suppose $g^*$ solves problem (3). By construction, $\Pi_k\Big(H\big(\phi(t_k^*, t_{-k}), \alpha\big), Y\Big) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k^{*p} = \Pi_k(g^*, Y) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} \tilde{t}_k^p(\alpha, g^*, t_{-k})$. Suppose $t_k^*$ does not solves problem (1). Then, $\Pi_k\Big(H\big(\phi(t'_k, t_{-k}), \alpha\big), Y\Big) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k^{\prime p} > \Pi_k\Big(H\big(\phi(t_k^*, t_{-k}), \alpha\big), Y\Big) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k^{*p}$ for some $t'_k \in \mathbb{R}_+^{n-K_\alpha}$. Let $H\big(\phi(t'_k, t_{-k}), \alpha\big) = g'$. Let $\tilde{t}_k(\alpha, g', t_{-k})$ be the solution of problem (2) for $\overline{g} = g'$. Then, $\Pi_k(g', Y) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} \tilde{t}_k^p(\alpha, g', t_{-k}) \geq \Pi_k(g', Y) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k'^{\,p}$. Suppose $g' = g^*$. Therefore, $\Pi_k(g^*, Y) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} \tilde{t}_k^p(\alpha, g^*, t_{-k}) > \Pi_k(g^*, Y) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} \tilde{t}_k^p(\alpha, g^*, t_{-k})$, which is a contradiction. Now, suppose that $g' \neq g^*$. Then, $\Pi_k(g', Y) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} \tilde{t}_k^p(\alpha, g', t_{-k}) > \Pi_k(g^*, Y) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} \tilde{t}_k^p(\alpha, g^*, t_{-k})$, which contradicts that $g^*$ solves problem (3). Therefore $\pi_k\big(\phi(t_k^*, t_{-k}), \alpha\big) \geq \pi_k\big(\phi(t'_k, t_{-k}), \alpha\big)$ for all $t'_k \in T_k(\alpha)$.

Now we prove the if part. Suppose $t_k^*$ is such that $\pi_k\big(\phi(t_k^*, t_{-k}), \alpha\big) \geq \pi_k\big(\phi(t'_k, t_{-k}), \alpha\big)$ for all $t'_k \in T_k(\alpha)$, i.e. $t_k^*$ solves problem (1). Suppose $t_k^*$ does not solve problem (2) for $\overline{g} = g^*$. Then, there exists $t'_k \in \mathbb{R}_+^{n-K_\alpha}$ such that $H\big(\phi(t'_k, t_{-k}), \alpha\big) = g^*$ and $\Pi_k(g^*, Y) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k^{\prime p} > \Pi_k(g^*, Y) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k^{*p}$. Therefore, $\Pi_k\Big(H\big(\phi(t'_k, t_{-k}), \alpha\big), Y\Big) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k^{\prime p} > \Pi_k\Big(H\big(\phi(t_k^*, t_{-k}), \alpha\big), Y\Big) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k^{*p}$, which contradicts $\pi_k\big(\phi(t_k^*, t_{-k}), \alpha\big) \geq \pi_k\big(\phi(t'_k, t_{-k}), \alpha\big)$ for all $t'_k \in T_k(\alpha)$. Thus, $t_k^*$ solves problem (2) for $\overline{g} = g^*$, i.e. $t_k^* = \tilde{t}_k^p(\alpha, g^*, t_{-k})$. Suppose $g^*$ does not solve problem (3). Then, there exists $g' \in \mathcal{X}_f(\alpha, t_{-k})$ such that $\Pi_k(g', Y) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} \tilde{t}_k^p(\alpha, g', t_{-k}) > \Pi_k(g^*, Y) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} \tilde{t}_k^p(\alpha, g^*, t_{-k})$. Let $t'_k = \tilde{t}_k^p(\alpha, g', t_{-k})$ and $H\big(\phi(t'_k, t_{-k}), \alpha\big) = g'$. Therefore, $\Pi_k\Big(H\big(\phi(t'_k, t_{-k}), \alpha\big), Y\Big) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k'^{\,p} > \Pi_k\Big(H\big(\phi(t_k^*, t_{-k}), \alpha\big), Y\Big) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t_k^{*p}$, which contradicts $\pi_k\big(\phi(t_k^*, t_{-k}), \alpha\big) \geq \pi_k\big(\phi(t'_k, t_{-k}), \alpha\big)$ for all $t'_k \in T_k(\alpha)$. □

**Proposition 2.** *For a fixed social structure $\alpha$, a poor agent receives strictly positive transfers only if she is in the direct diffusion network. That is, for a fixed $\alpha$, if in equilibrium $k \in \mathcal{K}(\alpha)$ allocates a strictly positive transfer to poor node $p \in \mathcal{P}(\alpha)$, then $p \in N^{DIF(\alpha)}$.*

This proof has four steps. In a first step we trivially show that if there are no ramified agents, then only agents in the direct diffusion path receive strictly positive transfers. Then, we show in three steps that the latter is also true when there exist ramified agents. First, we show that agents with no path to any rich agent under the ex-post production network $g^*$ receive zero transfers. Second, we show that the resource needs of ramified agents with a path to any rich node in $g^*$ are null. Lastly, we use Lemma 1 and Proposition 1 to show that these ramified agent receive zero transfers in equilibrium. We conclude that if an agent $p$ receives a strictly positive transfer, then $p$ is poor and is in the direct diffusion path.

Fix $\alpha$. The proof is trivial for $\alpha$ such that $N^{R(\alpha)} = \varnothing$ since $g^0 = DIF(\alpha)$. Now, suppose $N^{R(\alpha)} \neq \varnothing$. Let $\tau^*$ be any equilibrium transfer profile such that $t_k^{*^p} > 0$ for some $k \in \mathcal{K}(\alpha)$ and some $p \in \mathcal{P}(\alpha)$. Let $t^*$ be the aggregate transfer vector implied by $\tau^*$. Let $H(t^*, \alpha) = g^*$. Define $\mathcal{P}_0(\alpha, g^*) = \{p \in \mathcal{P}(\alpha) : \Theta_{pk}(g^*) = \varnothing \text{ for all } k \in \mathcal{K}(\alpha)\}$. We show first that $t_k^{*^p} > 0$ for some $k \in \mathcal{K}(\alpha)$ and some $p \in \mathcal{P}_0(\alpha, g^*)$ implies that $\tau^*$ does not solve problem (1) for some $k \in \mathcal{K}(\alpha)$.

Let $\mathcal{P}_1(\alpha, g^*) = \mathcal{P}(\alpha) - \mathcal{P}_0(\alpha, g^*)$, and suppose that $\tau^*(\alpha)$ is such that $t_k^{*^p} > 0$ for some $k \in \mathcal{K}(\alpha)$ and some $p \in \mathcal{P}_0(\alpha, g^*)$. For $k \in \mathcal{K}(\alpha)$, let $t'_k$ be a transfer vector such that $t_k^{'^p} = t_k^{*^p}$ for all $p \in \mathcal{P}_1(\alpha, g^*)$ and $t_k^{'^p} = 0$ for all $p \in \mathcal{P}_0(\alpha, g^*)$. Let $g' = g^0[N - \mathcal{P}_0(\alpha, g^*)]$. By the definition of $\mathcal{P}_0(\alpha, g^*)$, $\eta_k(g^*) = \eta_k(g')$ for all $k \in \mathcal{K}(\alpha)$. The definition of $\mathcal{P}_0(\alpha, g^*)$ also implies that $H\big(\phi(t'_k, t^*_{-k}), \alpha\big) = g^0[N - \mathcal{P}']$ for some $\mathcal{P}' \subseteq \mathcal{P}_0(\alpha, g^*)$. Therefore, $\eta_k(g^*) = \eta_k(g^0[N - \mathcal{P}'])$ for any $\mathcal{P}' \subseteq \mathcal{P}_0(\alpha, g^*)$. Then, $\eta_k\left(H\big(\phi(t'_k, t^*_{-k}), \alpha\big)\right) = \eta_k\left(H\big(\phi(t^*_k, t^*_{-k}), \alpha\big)\right)$ for all $k \in \mathcal{K}(\alpha)$. Therefore, the definition of $\Pi_l$ implies $\Pi_k(H\big(\phi(t^*_k, t^*_{-k}), \alpha\big), Y) = \Pi_k(H\big(\phi(t'_k, t^*_{-k}), \alpha\big), Y)$ for all $k \in \mathcal{K}(\alpha)$. Thus, $\Pi_k\Big(H\big(\phi(t'_k, t^*_{-k}), \alpha\big), Y\Big) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t'^p_k > \Pi_k\Big(H\big(\phi(t^*_k, t^*_{-k}), \alpha\big), Y\Big) - \epsilon_k - \sum_{p \in \mathcal{P}(\alpha)} t^{*p}_k$ for all $k \in \mathcal{K}(\alpha)$ such that $t_k^{*^p} > 0$. Then, $\tau^*$ such that $t_k^{*^p} > 0$ for some $k \in \mathcal{K}(\alpha)$ and some $p \in \mathcal{P}(\alpha)$ does not solve problem (1) for some $k \in \mathcal{K}(\alpha)$. Hence, if $\tau^*$ is an equilibrium profile, then $t^{*^p} = 0$ for all $p \in \mathcal{P}_0(\alpha, g^*)$.

Now we show that $t^{*^p} = 0$ for all $p \in N^{R(\alpha)} \cap \mathcal{P}_1(\alpha, g^*)$, which completes the proof. Suppose $\mathcal{P}_1(\alpha, g^*) = \varnothing$. Then, $t^{*^p} = 0$ for all $p \in \mathcal{P}(\alpha)$ which contradicts $\tau^*$ by definition. Suppose that $\mathcal{P}_1(\alpha, g^*) \neq \varnothing$. Then, we have to analyze two cases: $\tilde{\mathcal{P}}(\alpha, g^*) = \varnothing$ and $\tilde{\mathcal{P}}(\alpha, g^*) \neq \varnothing$.

First, let us suppose that $\tilde{\mathcal{P}}(\alpha, g^*) = \varnothing$. The definition of $\mathcal{P}_1(\alpha, g^*)$ and $DIF(\alpha)$ imply that $\mathcal{P}(\alpha) \cap N^{DIF(\alpha)} \subseteq \mathcal{P}_1(\alpha, g^*) \neq \varnothing$. Now we proceed to show that $\mathcal{P}(\alpha) \cap N^{DIF(\alpha)} = \mathcal{P}_1(\alpha, g^*)$. Suppose $\mathcal{P}(\alpha) \cap N^{DIF(\alpha)} \subset \mathcal{P}_1(\alpha, g^*)$. Then, there exists $\hat{p} \in \mathcal{P}_1(\alpha, g^*)$ such that $\hat{p} \notin \mathcal{P}(\alpha) \cap N^{DIF(\alpha)}$. Then, $\hat{p} \in \mathcal{P}(\alpha)$ implies that $\hat{p} \in N^{R(\alpha)}$. Also, $\hat{p} \in \mathcal{P}_1(\alpha, g^*)$ implies that $\Theta_{k\hat{p}}(g^*) \neq \varnothing$ for some $k \in \mathcal{K}(\alpha)$. Therefore, (i) $\Theta_{\hat{p}\tilde{p}}(g^*) \neq \varnothing$ for some $\tilde{p} \in \mathcal{P}(\alpha) \cap N^{DIF(\alpha)}$, or (ii) $\hat{p} \in \eta_k(g^*)$ for some $k \in \mathcal{K}(\alpha)$. Suppose condition (i). Then there exists $\theta_{\tilde{p}\hat{p}} \in \Theta_{\tilde{p}\hat{p}}(g^*)$ such that $\tilde{p} \in N^{DIF(\alpha)}$. Then $\tilde{\mathcal{P}}(\alpha, g^*) = \varnothing$ implies that $\hat{p} \notin N^{R(\alpha)}$, which contradicts $\hat{p} \in N^{R(\alpha)}$. Suppose condition (ii). Then, there are two cases. First, $\hat{p} \in \mathcal{S}^1(\alpha)$. Then there exists $\theta_{kj} \in \Theta_{kj}(g^*)$ for $k \in \mathcal{K}(\alpha)$ and $j \in \mathcal{S}^1(\alpha)$ such that $\hat{p} = j$ and $\hat{p} \in N^{\theta_{kj}}$, which contradicts $\hat{p} \in N^{R(\alpha)}$. Second, $\hat{p} \notin \mathcal{S}^1(\alpha)$. Then, $\hat{p} \in \mathcal{P}(\alpha)$ implies that $\hat{p} \in \mathcal{S}^i(\alpha)$ for $i > 1$. Thus, the definition of layer implies that there exists some $j \in \mathcal{S}^1(\alpha)$ and $j \neq \hat{p}$ such that $\theta_{kj} \in \Theta_{kj}(g^*)$ for $k \in \mathcal{K}(\alpha)$ and $\hat{p} \in N^{\theta_{kj}}$, which also contradicts $\hat{p} \in N^{R(\alpha)}$. Hence, $\mathcal{P}(\alpha) \cap N^{DIF(\alpha)} = \mathcal{P}_1(\alpha, g^*)$. The assumption $\tilde{\mathcal{P}}(\alpha, g^*) = \varnothing$ implies that $\mathcal{P}(\alpha) \cap N^{DIF(\alpha)} \cap N^{R(\alpha)} = \varnothing$. Therefore, $\mathcal{P}_1(\alpha, g^*) \cap N^{R(\alpha)} = \varnothing$. In the first part of the proof we showed that $\tau^*$ implies that $t^{*^p} = 0$ for all $p \in \mathcal{P}_0(\alpha, g^*)$. Therefore, the definition of $\mathcal{P}_1(\alpha, g^*)$ implies that if $t^{*^p} > 0$, then $p \in N^{DIF(\alpha)}$.

Now, suppose $\tilde{\mathcal{P}}(\alpha, g^*) \neq \varnothing$. Pick any $p' \in \tilde{\mathcal{P}}(\alpha, g^*)$. The definition of $N_s$-subnetwork implies that $\hat{\mathcal{P}}(\alpha, g^*, p') \subseteq \hat{\mathcal{P}}(\alpha, g^0, p')$. Now we proceed to show that $\hat{\mathcal{P}}(\alpha, g^*, p') = \hat{\mathcal{P}}(\alpha, g^0, p')$. Suppose $\hat{\mathcal{P}}(\alpha, g^*, p') \subset \hat{\mathcal{P}}(\alpha, g^0, p')$. Then, there exists $l' \in \hat{\mathcal{P}}(\alpha, g^0, p')$ such that $l' \notin \hat{\mathcal{P}}(\alpha, g^*, p')$. By Lemma 6, $\Pi_p(g^0[\hat{\mathcal{P}}(\alpha, g^0, p') \cup \{p'\}], Y) - \epsilon_p \geq 0$ for all $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$. Lemma 6 also implies that $\eta_p(g^0) = \eta_p(g^0[\hat{\mathcal{P}}(\alpha, g^0, p') \cup \{p'\}])$ for all $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$. Then, $p' \in \tilde{\mathcal{P}}(\alpha, g^*)$ and $l' \notin \hat{\mathcal{P}}(\alpha, g^*, p')$ imply that there exists some $l'' \in \hat{\mathcal{P}}(\alpha, g^0, p')$ such that $l'' \neq l'$ and $\Pi_{l''}(g^0[\hat{\mathcal{P}}(\alpha, g^0, p') \cup \{p'\}], Y) - \epsilon_{l''} + t^{*^{l''}} < 0$, which contradicts Lemma 5, or contradicts the definition of $T_k(\alpha)$ for some $k \in \mathcal{K}(\alpha)$. Hence, $\hat{\mathcal{P}}(\alpha, g^*, p') = \hat{\mathcal{P}}(\alpha, g^0, p')$.

Therefore, $\Pi_p(g^0[\hat{\mathcal{P}}(\alpha, g^*, p')], Y) - \epsilon_p \geq 0$ for all $p \in \hat{\mathcal{P}}(\alpha, g^*, p')$. Then, the definition of $r_p$ implies that $r_p(\alpha, g^*, 0) = 0$ for all $p \in \hat{\mathcal{P}}(\alpha, g^*, p')$. To complete the proof for the case $\tilde{\mathcal{P}}(\alpha, g^*) \neq \varnothing$, we show as an intermediary step that $\bigcup_{p' \in \tilde{\mathcal{P}}(\alpha, g^*)} \hat{\mathcal{P}}(\alpha, g^*, p') = N^{R(\alpha)} \cap \mathcal{P}_1(\alpha, g^*)$, which, we will show, implies $r_p(\alpha, g^*, 0) = 0$ for all $p \in N^{R(\alpha)} \cap \mathcal{P}_1(\alpha, g^*)$. The definition of $\hat{\mathcal{P}}(\alpha, g^*, p')$ implies that $\bigcup_{p' \in \tilde{\mathcal{P}}(\alpha, g^*)} \hat{\mathcal{P}}(\alpha, g^*, p') \subseteq N^{R(\alpha)}$. The definition of $\hat{\mathcal{P}}(\alpha, g^*, p')$ also implies that there exists $\theta_{pp'} \in \Theta_{pp'}(g^0)$ for all $p \in \hat{\mathcal{P}}(\alpha, g^*, p')$ and $p' \in \tilde{\mathcal{P}}(\alpha, g^*)$. The definition of $\tilde{\mathcal{P}}(\alpha, g^*)$ implies that that there exists $\theta_{p'k} \in \Theta_{p'k}(g^0)$ for $p' \in \tilde{\mathcal{P}}(\alpha, g^*)$ and $k \in \mathcal{K}(\alpha)$ such that $N^{\theta_{p'k}} \subseteq N^{DIF(\alpha)}$. Pick any $\theta_{pp'} \in \Theta_{pp'}(g^0)$ and $\theta_{p'k} \in \Theta_{p'k}(g^0)$ such that $N^{\theta_{p'k}} \subseteq N^{DIF(\alpha)}$. Suppose that there is no path $\theta_{pp'} + \theta_{p'k} \in \Theta_{pk}(g^0)$. Then, the definition of path implies that there exists $z \in N^{\theta_{p'k}}$ and $z \in N^{\theta_{pp'}}$ such that $z \neq p'$. The definition of path also implies that $d_{pz}(\theta_{pp'}) < d_{pp'}(\theta_{pp'})$. Moreover $z \in N^{\theta_{p'k}}$ implies that $z \in N^{DIF(\alpha)}$. Then $d_{pz}(\theta_{pp'}) < d_{pp'}(\theta_{pp'})$ and $z \in N^{DIF(\alpha)}$ contradict $p \in \hat{\mathcal{P}}(\alpha, g^*, p')$. Therefore, $\bigcup_{p' \in \tilde{\mathcal{P}}(\alpha,g^*)} \hat{\mathcal{P}}(\alpha, g^*, p') = \mathcal{P}_1(\alpha, g^*)$. Thus, $\bigcup_{p' \in \tilde{\mathcal{P}}(\alpha,g^*)} \hat{\mathcal{P}}(\alpha, g^*, p') = N^{R(\alpha)} \cap \mathcal{P}_1(\alpha, g^*)$.

Hence, $r_p(\alpha, g^*, 0) = 0$ for all $p \in N^{R(\alpha)} \cap \mathcal{P}_1(\alpha, g^*)$. The latter implies that, by Lemma 1, $\tilde{t}^p_k(\alpha, g^*, t^*_{-k}) = 0$ for each $k \in \mathcal{K}(\alpha)$, and each $p \in N^{R(\alpha)} \cap \mathcal{P}_1(\alpha, g^*)$. Therefore, Proposition 1 implies that if $\tau^*$ is an equilibrium transfer profile, then $t^{*^p} = 0$ for each $p \in \left(N^{R(\alpha)} \cap \mathcal{P}_1(\alpha, g^*)\right) \cup \mathcal{P}_0(\alpha, g^*)$. Therefore, by construction, $\mathcal{P}(\alpha) - \left(\left(N^{R(\alpha)} \cap \mathcal{P}_1(\alpha, g^*)\right) \cup \mathcal{P}_0(\alpha, g^*)\right) = N^{DIF(\alpha)} \cap \mathcal{P}(\alpha)$ which implies that if $t^{*^p} > 0$, then $p \in N^{DIF(\alpha)}$.

**Proposition 3.** *In a multilayer social structure topology, there is at least one poor node that does not receive positive transfers in equilibrium. That is, if $\omega(\alpha)$ is multilayer, then $\#U(g_T) < n - K_\alpha$ for all $g_T \in \Phi(\alpha)$.*

*Proof.* Pick any $\alpha \in \mathcal{A}$ such that $\omega(\alpha)$ is multilayer. Let $t^*$ be an equilibrium aggregate transfer vector. Let $g^*_T$ be a giving network implied by $t^*$. By construction $\#U(g^*_T) \leq n - K_\alpha$. Suppose $\#U(g^*_T) = n - K_\alpha$. Then, $t^*$ is such that $t^{*^p} > 0$ for all $p \in \mathcal{P}(\alpha)$. We first show that $t^{*^p} > 0$ for all $p \in \mathcal{P}(\alpha)$ implies $H(\alpha, t^*) = g^0$. Then we use Lemma 1 to complete the proof.

Suppose $g^0 = \varnothing$. Then $\mathcal{X}(\alpha) = \{g^0\}$. Thus $H(\alpha, t^*) = g^0$. Now suppose that $g^0 \neq \varnothing$ and $H(\alpha, t^*) = \hat{g} \in \mathcal{X}(\alpha)$ such that $\hat{g} \neq g^0$. Then $\hat{g} \neq g^0$ implies that there exists $\hat{p} \in \mathcal{P}(\alpha)$ such that $\eta_{\hat{p}}(g^0) \neq \varnothing$ and $\eta_{\hat{p}}(\hat{g}) = \varnothing$. Hence, Lemma 1 implies that $\tilde{t}^{\hat{p}}_k(\alpha, \hat{g}, t^*_{-k}) = 0$ for all $k \in \mathcal{K}(\alpha)$. Therefore, $t^{*^p} > 0$ for all $p \in \mathcal{P}(\alpha)$ does not solve problem (2) for $\overline{g} = \hat{g}$. Thus, Proposition 1 implies that $t^{*^p} > 0$ for all $p \in \mathcal{P}(\alpha)$ does not solve problem (1), which contradicts the definition of $t^*$. Therefore, $H(\alpha, t^*) = g^0$. Suppose $\eta_p(g^0) = \varnothing$ for some $p \in \mathcal{P}(\alpha)$. Then, Lemma 1 implies that $\tilde{t}^{\hat{p}}_k(\alpha, g^0, t^*_{-k}) = 0$ for all $k \in \mathcal{K}(\alpha)$. Therefore, $t^{*^p} > 0$ for all $p \in \mathcal{P}(\alpha)$ does not solve problem (2) for $\overline{g} = \hat{g}$. Thus, Proposition 1 implies that $t^{*^p} > 0$ for all $p \in \mathcal{P}(\alpha)$ does not solve problem (1), which contradicts the definition of $t^*$. Suppose now that $\eta_p(g^0) \neq \varnothing$ for all $p \in \mathcal{P}(\alpha)$. By the definition of layer, $\Pi_p(g^0, Y) - \epsilon_p \geq 0$ for all $p \notin \mathcal{S}^1(\alpha)$. Thus, $r_p(\alpha, g^0, 0) = 0$ for all $p \notin \mathcal{S}^1(\alpha)$. Then, Lemma 1 implies that $\tilde{t}^{\hat{p}}_k(\alpha, g^0, t^*_{-k}) = 0$ for all $p \notin \mathcal{S}^1(\alpha)$ and all $k \in \mathcal{K}(\alpha)$. Therefore, $t^{*^p} > 0$ for all $p \in \mathcal{P}(\alpha)$ does not solve problem (2) for $\overline{g} = \hat{g}$. Thus, Proposition 1 implies that

$t^{*^p} > 0$ for all $p \in \mathcal{P}(\alpha)$ does not solve problem (1), which contradicts the definition of $t^*$. Hence, $\#U(g_T) < n - K_\alpha$ for all $g_T \in \Phi(\alpha)$. □

**Proposition 4.** *In social structures where all the poor agents are either disconnected or directly connected with the rich agents in the ex ante production network, the equilibrium giving network is such that there are no links that do not exist in the ex ante production network. That is, if $\alpha \in \tilde{\mathcal{A}}$, then $G^{g_T} \subseteq G^{g^0}$ for $g_T \in \Phi(\alpha)$.*

*Proof.* Pick any $\alpha \in \tilde{\mathcal{A}}$. Let $\tau^*$ be an equilibrium transfer profile. Let $t^*$ be the equilibrium aggregate transfer vector implied by $\tau^*$. Let $g_T^*$ be any equilibrium giving network implied by $t^*$. First, suppose $g_T^* = g^\varnothing$. Then, $G^{g_T^*} = \varnothing$. Therefore, $G^{g_T^*} \subseteq G^{g^0}$. Now, suppose that $g_T^* \neq g^\varnothing$ and $G^{g_T^*} \nsubseteq G^{g^0}$. Then, $kp \in G^{g_T^*}$ and $kp \notin G^{g^0}$ for some $k \in \mathcal{K}(\alpha)$ and some $p \in \mathcal{P}(\alpha)$. Pick any $\tilde{k} \in \mathcal{K}(\alpha)$ such that $\tilde{k}p \in G^{g_T^*}$ and $\tilde{k}p \notin G^{g^0}$ for some $p \in \mathcal{P}(\alpha)$. Let $\tilde{\mathcal{P}} = \{p \in \mathcal{P}(\alpha) : \tilde{k}p \in G^{g_T^*}$ and $\tilde{k}p \notin G^{g^0}\}$. The definition of giving network implies that $t_{\tilde{k}}^{*^p} > 0$ for all $p \in \tilde{\mathcal{P}}$. Let $t'_{\tilde{k}}$ be a transfer vector such that $t_{\tilde{k}}^{\prime p} = t_{\tilde{k}}^{*^p}$ for all $p \in \mathcal{P}(\alpha) - \tilde{\mathcal{P}}$ and $t_{\tilde{k}}^{\prime p} = 0$ for all $p \in \tilde{\mathcal{P}}$. By the definition of $\tilde{\mathcal{A}}$, $d_{\tilde{k}p}(g^0) = \infty$ for all $p \in \tilde{\mathcal{P}}$. Hence, $\Theta_{\tilde{k}p}(g^0) = \varnothing$ for all $p \in \tilde{\mathcal{P}}$. Thus, $\eta_{\tilde{k}}\Big(H\big(\phi(t'_{\tilde{k}}, t^*_{-\tilde{k}}), \alpha\big)\Big) = \eta_{\tilde{k}}\Big(H\big(\phi(t^*_{\tilde{k}}, t^*_{-\tilde{k}}), \alpha\big)\Big)$. Therefore, the definition of $\Pi_l$ implies $\Pi_{\tilde{k}}(H\big(\phi(t^*_{\tilde{k}}, t^*_{-\tilde{k}}), \alpha\big), Y) = \Pi_{\tilde{k}}(H\big(\phi(t'_{\tilde{k}}, t^*_{-\tilde{k}}), \alpha\big), Y)$. Thus, $\Pi_{\tilde{k}}\Big(H\big(\phi(t'_{\tilde{k}}, t^*_{-\tilde{k}}), \alpha\big), Y\Big) - \epsilon_{\tilde{k}} - \sum_{p \in \mathcal{P}(\alpha)} t_{\tilde{k}}^{\prime p} > \Pi_{\tilde{k}}\Big(H\big(\phi(t^*_{\tilde{k}}, t^*_{-\tilde{k}}), \alpha\big), Y\Big) - \epsilon_{\tilde{k}} - \sum_{p \in \mathcal{P}(\alpha)} t_{\tilde{k}}^{*p}$. Then, $\tau^*$ such that $t_{\tilde{k}}^{*^p} > 0$ for all $p \in \tilde{\mathcal{P}}$ does not solve problem (1) for $\tilde{k} \in \mathcal{K}(\alpha)$. Hence, $G^{g_T^*} \subseteq G^{g^0}$. □

**Proposition 5.** *In the set of social structures where there exists at least one indirect connections between a poor agent and a rich agent, there exists some social structure such that some of its equilibrium giving network contains a link between agents that are not directly connected in the ex ante production network. That is, there exists $\alpha \in \mathcal{A}$ such that $G^{g_T} \nsubseteq G^{g^0}$ for $g_T \in \Phi(\alpha)$.*

*Proof.* Pick $\tilde{\alpha} \in \mathcal{A}$ such that $g^0 = \big(\{1, ..., \tilde{n} - 1, \tilde{n}\}, \{k1, 12, 23, ..., \tilde{n} - 1\tilde{n}\}\big)$, $\mathcal{K}(\tilde{\alpha}) = \{k\}$, and $\tilde{n} + 1 = n$. Suppose

(i) $y_i^{i+1} > \epsilon_i + \underline{y} - y_i^i - y_i^{i-1}$ for all $i \in \{1, ..., \tilde{n} - 1\}$

(ii) $0 < \epsilon_{\tilde{n}} + \underline{y} - y_{\tilde{n}}^{\tilde{n}} - y_{\tilde{n}}^{\tilde{n}-1} < \epsilon_i + \underline{y} - y_i^i - y_i^{i-1} < y_k^1$ for all $i \in \{1, ..., \tilde{n} - 1\}$

(iii) $y_k^k - \underline{y} - \epsilon_k > \epsilon_{\tilde{n}} + \underline{y} - y_{\tilde{n}}^{\tilde{n}} - y_{\tilde{n}}^{\tilde{n}-1}$.

The definitions of $g^0$ and $\mathcal{K}(\tilde{\alpha})$ imply that $\tilde{\alpha} \in \tilde{\mathcal{A}}^c$. Moreover $Y \in \mathbb{R}_+^{n^2}$ and $\epsilon \in \mathbb{R}^n$ imply that $\tilde{\alpha}$ satisfies conditions (i), (ii), and (iii). We now derive the solution to problem (2) for $k$ in $\tilde{\alpha}$. We then derive the solution of problem (3) for $k$ in $\tilde{\alpha}$. Lastly, we use Proposition 1 to complete the proof.

Condition (i) implies that $r_i(\tilde{\alpha}, g^0[N - \{i+2, ..., \tilde{n}\}], 0) = 0$ for all $i \in \{1, ..., \tilde{n}-2\}$ and $r_{\tilde{n}-1}(\tilde{\alpha}, g^0, 0) = 0$. Condition (ii) implies that $r_i(\tilde{\alpha}, g^0[N - \{i+1, ..., \tilde{n}\}], 0) = \epsilon_i + \underline{y} - y_i^i - y_i^{i-1}$ for all $i \in \{1, ..., \tilde{n}-1\}$ and $r_{\tilde{n}}(\tilde{\alpha}, g^0, 0) = \epsilon_{\tilde{n}} + \underline{y} - y_{\tilde{n}}^{\tilde{n}} - y_{\tilde{n}}^{\tilde{n}-1}$. Let $\tilde{t}_k(\alpha, \overline{g}, 0)$ be a $k$'s solution to problem (2) for $\overline{g} \in \mathcal{X}(\tilde{\alpha}, 0)$ (Las funciones $\mathcal{X}(\tilde{\alpha}, 0)$ y $\mathcal{X}(\tilde{\alpha})$ estan bien definidas, pero la notacion es mala. Hay que cambiarla). In section 2.6 we show that $\tilde{t}_k(\alpha, \overline{g}, 0)$ exists for each $\overline{g} \in \mathcal{X}(\tilde{\alpha}, 0)$. Conditions (i) and (ii) imply that $\mathcal{X}(\tilde{\alpha}, 0) = \{\bigcup_{i \in \{1,..,\tilde{n}-1\}} g^0[N - \{i+1, ..., \tilde{n}\}] \cup g^0 \cup g^{\varnothing}\}$. Then, Lemma 1 implies that $\tilde{t}_k(\alpha, \overline{g}, 0)$ is such that

(a) $\tilde{t}_k^i(\alpha, \overline{g}, 0) = \epsilon_i + \underline{y} - y_i^i - y_i^{i-1}$ and $\tilde{t}_k^{i'}(\alpha, \overline{g}, 0) = 0$ for all $i' \neq i$ and $i' \in \mathcal{P}(\tilde{\alpha})$, for $i \in \{1, .., \tilde{n}-1\}$ and $\overline{g} = g^0[N - \{i+1, ..., \tilde{n}\}]$.

(b) $\tilde{t}_k^{\tilde{n}}(\alpha, \overline{g}, 0) = \epsilon_{\tilde{n}} + \underline{y} - y_{\tilde{n}}^{\tilde{n}} - y_{\tilde{n}}^{\tilde{n}-1}$ and $\tilde{t}_k^{i'}(\alpha, \overline{g}, 0) = 0$ for all $i' \in \{1, .., \tilde{n}-1\}$ and $\overline{g} = g^0$

(c) $\tilde{t}_k^i(\alpha, \overline{g}, 0) = 0$ for all $i \in \{1, ..., \tilde{n}\}$ and $\overline{g} = g^{\varnothing}$.

Let $g^*$ be a $k$'s solution to problem (3). We showed in section 2.6 that $g^*$ exists. Now, we prove that $g^* = g^0$. Condition (iii) implies that $\mathcal{X}_f(\tilde{\alpha}, 0) = \mathcal{X}(\tilde{\alpha}, 0)$. The definition of $g^0$ implies that $\eta_k(\overline{g}) = \{1\}$ for all $\overline{g} \in \mathcal{X}_f(\tilde{\alpha}, 0)$ such that $\overline{g} \neq g^{\varnothing}$, and $\eta_k(\overline{g}) = \varnothing$ for $\overline{g} = g^{\varnothing}$. Then, implications (a) through (c) and condition (ii) imply that $g^* = g^0$.

Thus Proposition 1 implies that $t_k^{*\tilde{n}} = \epsilon_{\tilde{n}} + \underline{y} - y_{\tilde{n}}^{\tilde{n}} - y_{\tilde{n}}^{\tilde{n}-1}$ and $t_k^{*i} = 0$ for all $i \in \{1, .., \tilde{n}-1\}$ solve problem (1) for $k \in \mathcal{K}(\tilde{\alpha})$. Then $\mathcal{K}(\alpha) = \{k\}$ implies that $t^*$ such that $t_k^{*\tilde{n}} = \epsilon_{\tilde{n}} + \underline{y} - y_{\tilde{n}}^{\tilde{n}} - y_{\tilde{n}}^{\tilde{n}-1}$ and $t_k^{*i} = 0$ for all $i \in \{1, .., \tilde{n}-1\}$ is an equilibrium transfer vector. Hence there exists $\tilde{\alpha} \in \tilde{\mathcal{A}}^c$ such that $G^{g_T} \notin G^{g^0}$ for $g_T \in \Phi(\alpha)$.

□

**Proposition 6.** *For each social structure there exists a social structure with a different underlying ex ante production network such that both induce the same equilibrium giving network. Formally, pick any $\alpha \in \hat{\mathcal{A}}$. Suppose $g_T^* \in \Phi(\alpha)$ exists. Then, there exists $\hat{\alpha} \in \hat{\mathcal{A}}$ such that $g^0 \neq \hat{g}^0$, $g_T^* = \hat{g}_T$, and $\hat{g}_T \in \Phi(\hat{\alpha})$.*

*Proof.* The proof has two parts. We first prove Proposition 6 for the case $g_T^* \neq g^{\varnothing}$. To do so, we first pick an economy $\hat{\alpha}$ which ex ante production network is by construction different from the ex ante production network in $\alpha$. We show that $\hat{\alpha}$ induces the same giving network as $\alpha$. Then, we prove the statement of Proposition 6 for $g_T^* = g^{\varnothing}$ following analogous steps as for the case $g_T^* \neq g^{\varnothing}$.

Pick $\alpha \in \hat{\mathcal{A}}$. Suppose $\Phi(\alpha) \neq \varnothing$. Pick any $g_T^* \in \Phi(\alpha)$. Suppose $g_T^* \neq g^{\varnothing}$. Let $\hat{\alpha} \in \hat{\mathcal{A}}$ be such that the ex ante production network of $\hat{\alpha}$ is $\hat{g}^0 = (N, \hat{G})$, and $\omega(\hat{\alpha}) = \big(\{N - E(g_T^*)\}, E(g_T^*), \hat{G}\big)$; where

$$\hat{G} = \begin{cases} G^{g_T^*} & \text{if } \; g_T^* \neq g^0 \\ G^{g_T^*} \cup \{k'k''\} & \text{if } \; g_T^* = g^0 \text{ and } \#E(g_T^*) > 1 \\ G^{g_T^*} \cup \{p_0 p_1\} & \text{if } \; g_T^* = g^0 \text{ and } \#E(g_T^*) = 1, \end{cases}$$

with $k', k'' \in \mathcal{K}(\hat{\alpha})$, $p_0, p_1 \in \mathcal{P}(\hat{\alpha})$, and

(i) $r_p(\hat{\alpha}, \hat{g}^0, 0) = c > 0$ for all $p \in \mathcal{P}(\hat{\alpha})$,

(ii) $y_k^p > c$ for each $k \in \mathcal{K}(\hat{\alpha})$ and all $p \in \mathcal{P}(\hat{\alpha}) \cap \eta_k(\hat{g}^0)$,

(iii) $\Pi_k\left(g^{\mathcal{K}(\hat{\alpha})}, Y\right) - \epsilon_k > \#\mathcal{P}(\hat{\alpha}) \times c$ for all $k \in \mathcal{K}(\hat{\alpha})$, and

(iv) $y_p^{p'} = 0$ for $p, p' \in \mathcal{P}(\hat{\alpha})$ such that $pp' \in \hat{G}$ and $\{p, p'\} \cap \bigcup_{k \in \mathcal{K}(\hat{\alpha})} \eta_k(\hat{g}^0) = \{p\}$.

The definitions of $\Pi_k\left(g^{\mathcal{K}(\hat{\alpha})}, Y\right)$ and $r_p(\hat{\alpha}, \hat{g}^0, 0)$ imply that conditions (i) through (iv) are independent. Moreover, the definitions of $\Pi_k\left(g^{\mathcal{K}}(\hat{\alpha}), Y\right)$ and $r_p(\hat{\alpha}, \hat{g}^0, 0)$, and $Y \in \mathbb{R}_+^{n^2}$ and $\epsilon \in \mathbb{R}^n$ imply that there exists $\hat{\alpha}$ such that conditions (i) to (iv) are satisfied.

Consider a transfer profile $\hat{\tau}$ such that for each $k \in \mathcal{K}(\hat{\alpha})$, let $\hat{t}_k^p = c / \#(\eta_p(\hat{g}^0) \cap \mathcal{K}(\hat{\alpha}))$ for all $p \in \mathcal{P}(\hat{\alpha}) \cap \eta_k(\hat{g}^0)$ and $\hat{t}_k^p = 0$ for all $p \in \mathcal{P}(\hat{\alpha}) \cap [\eta_k(\hat{g}^0)]^c$, which implies the aggregate transfer vector $\hat{t}$. Let $\hat{g}_T$ be the giving network implied by $\hat{\tau}$. Then, the definitions of giving network and $\hat{G}$ imply that $\hat{g}_T = g_T^*$. By the construction of $\hat{G}$, $\hat{g}^0 \neq g^0$. Now we prove that $\hat{\tau}$ is an equilibrium transfer profile.

First, condition (iii) implies that $\hat{\tau}$ is feasible. Suppose $\tau'$ is an equilibrium transfer profile such that $\tau' \neq \hat{\tau}$. Let $t'$ be the aggregate transfer vector implied by $\tau'$. Then $t'_k \neq \hat{t}_k$ for some $k \in \mathcal{K}(\hat{\alpha})$. Pick any $k \in \mathcal{K}(\hat{\alpha})$ such that $t'_k \neq \hat{t}_k$ and denote this agent by $\tilde{k}$. Suppose $t'^{p}_{\tilde{k}} > 0$ for some $p \in \mathcal{P}(\hat{\alpha}) \cap [\eta_{\tilde{k}}(\hat{g}^0)]^c$ and $t'^{p}_{\tilde{k}} = \hat{t}^p_{\tilde{k}}$ otherwise. Then, the construction of $\tau'$ and condition (i) implies that $t'^p_{\tilde{k}} + \hat{t}^p_{\tilde{k}} = r_p(\hat{\alpha}, \hat{g}^0, 0)$ for all $p \in \mathcal{P}(\hat{\alpha}) \cap \eta_{\tilde{k}}(\hat{g}^0)$. Then, $\eta_{\tilde{k}}\Big(H\big(\phi(t'_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big)\Big) = \eta_{\tilde{k}}\Big(H\big(\phi(\hat{t}_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big)\Big)$. Therefore, the definition of $\Pi_l$ implies $\Pi_{\tilde{k}}(H\big(\phi(t'_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big), Y) = \Pi_{\tilde{k}}(H\big(\phi(\hat{t}_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big), Y)$. Thus, $\Pi_{\tilde{k}}\Big(H\big(\phi(t'_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big), Y\Big) - \epsilon_{\tilde{k}} - \sum\limits_{p \in \mathcal{P}(\alpha)} t'^{p}_{\tilde{k}} < \Pi_{\tilde{k}}\Big(H\big(\phi(\hat{t}_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big), Y\Big) - \epsilon_{\tilde{k}} - \sum\limits_{p \in \mathcal{P}(\alpha)} \hat{t}^p_{\tilde{k}}$. Then, $\tau'$ does not solve problem (1) for $\tilde{k} \in \mathcal{K}(\alpha)$.

Suppose that $t'^{p}_{\tilde{k}} > c/\#(\eta_p(\hat{g}^0) \cap \mathcal{K}(\hat{\alpha}))$ for some $p \in \mathcal{P}(\hat{\alpha}) \cap \eta_{\tilde{k}}(\hat{g}^0)$ and $t'^{p}_{\tilde{k}} = \hat{t}^p_{\tilde{k}}$ otherwise. Then, the construction of $\tau'$ and condition (i) implies that $t'^{p}_{\tilde{k}} + \hat{t}^p_{-\tilde{k}} > r_p(\hat{\alpha}, \hat{g}^0, 0)$ for some $p \in \mathcal{P}(\hat{\alpha}) \cap \eta_{\tilde{k}}(\hat{g}^0)$ and $t'^{p}_{\tilde{k}} + \hat{t}^p_{-\tilde{k}} \geq r_p(\hat{\alpha}, \hat{g}^0, 0)$ for all $p \in \mathcal{P}(\hat{\alpha}) \cap \eta_{\tilde{k}}(\hat{g}^0)$. Then, $\eta_{\tilde{k}}\Big(H\big(\phi(t'_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big)\Big) = \eta_{\tilde{k}}\Big(H\big(\phi(\hat{t}_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big)\Big)$. Therefore, the definition of $\Pi_l$ implies
$\Pi_{\tilde{k}}(H\big(\phi(t'_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big), Y) = \Pi_{\tilde{k}}(H\big(\phi(\hat{t}_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big), Y)$. Thus, $\Pi_{\tilde{k}}\Big(H\big(\phi(t'_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big), Y\Big) - \epsilon_{\tilde{k}} - \sum\limits_{p \in \mathcal{P}(\alpha)} t'^{p}_{\tilde{k}} < \Pi_{\tilde{k}}\Big(H\big(\phi(\hat{t}_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big), Y\Big) - \epsilon_{\tilde{k}} - \sum\limits_{p \in \mathcal{P}(\alpha)} \hat{t}^p_{\tilde{k}}$. Then, $\tau'$ does not solve problem (1) for $\tilde{k} \in \mathcal{K}(\alpha)$.

Suppose now that $t'^{p}_{\tilde{k}} = \tilde{c}$ such that $0 \leq \tilde{c} < c/\#(\eta_p(\hat{g}^0) \cap \mathcal{K}(\hat{\alpha}))$ for some $p \in \mathcal{P}(\hat{\alpha}) \cap \eta_{\tilde{k}}(\hat{g}^0)$ and $t'^{p}_{\tilde{k}} = \hat{t}^p_{\tilde{k}}$ otherwise. The construction of $\tau'$ and condition (i) implies that $t'^{p}_{\tilde{k}} + \hat{t}^p_{-\tilde{k}} < r_p(\hat{\alpha}, \hat{g}^0, 0)$ for some $p \in \mathcal{P}(\hat{\alpha}) \cap \eta_{\tilde{k}}(\hat{g}^0)$, and $t'^{p}_{\tilde{k}} + \hat{t}^p_{-\tilde{k}} \leq r_p(\hat{\alpha}, \hat{g}^0, 0)$ for all $p \in \mathcal{P}(\hat{\alpha}) \cap \eta_{\tilde{k}}(\hat{g}^0)$. Let $\mathcal{P}' = \{p \in \mathcal{P}(\hat{\alpha}) : t'^{p}_{\tilde{k}} + \hat{t}^p_{-\tilde{k}} < r_p(\hat{\alpha}, \hat{g}^0, 0)\}$. The construction of $\hat{G}$ and condition (i) also imply that there exists $\hat{\alpha}$ such that $\hat{G} = G^{g^*_T} \cup \{p_0 p_1\}$, $p_0, p_1 \in \mathcal{P}(\hat{\alpha})$, and $\{p_0, p_1\} \cap \bigcup\limits_{k \in \mathcal{K}(\hat{\alpha})} \eta_k(\hat{g}^0) = \{p_0, p_1\}$. In the latter case, $r_{p_0}(\hat{\alpha}, \hat{g}^0[N - p_1], 0) < c$ and $r_{p_1}(\hat{\alpha}, \hat{g}^0[N - p_0], 0) < c$, which implies that there exists $\mathcal{P}'' = \eta_{\tilde{k}}\Big(H\big(\phi(t'_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big)\Big) - \eta_{\tilde{k}}\Big(H\big(\phi(\hat{t}_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big)\Big)$ such that $\mathcal{P}'' \subseteq \mathcal{P}(\hat{\alpha})$ and $\mathcal{P}' \subseteq \mathcal{P}''$. Therefore, the definition of $\Pi_l$ implies $\Pi_{\tilde{k}}(H\big(\phi(\hat{t}_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big), Y) - \Pi_{\tilde{k}}(H\big(\phi(t'_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big), Y) = \sum_{p \in \mathcal{P}''} y^p_{\tilde{k}}$. By the construction of $\tau'$, $\sum_{p \in \mathcal{P}(\hat{\alpha})} \hat{t}^p_k - \sum_{p \in \mathcal{P}(\hat{\alpha})} t'^p_k = \sum_{p \in \mathcal{P}'} c/\#(\eta_p(\hat{g}^0) \cap \mathcal{K}(\hat{\alpha})) - \sum_{p \in \mathcal{P}'} \tilde{c}$. Then, condition (ii) implies that $\Pi_{\tilde{k}}\Big(H\big(\phi(t'_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big), Y\Big) - \epsilon_{\tilde{k}} - \sum\limits_{p \in \mathcal{P}(\alpha)} t'^{p}_{\tilde{k}} < \Pi_{\tilde{k}}\Big(H\big(\phi(\hat{t}_{\tilde{k}}, \hat{t}_{-\tilde{k}}), \alpha\big), Y\Big) - \epsilon_{\tilde{k}} - \sum\limits_{p \in \mathcal{P}(\alpha)} \hat{t}^p_{\tilde{k}}$. Then, $\tau'$ does not solve problem (1) for $\tilde{k} \in \mathcal{K}(\alpha)$.

Lastly, suppose that $t'^{p}_{\tilde{k}} > 0$ for some $p \in \mathcal{P}(\hat{\alpha}) \cap [\eta_{\tilde{k}}(\hat{g}^0)]^c$ and $t'^p_{\tilde{k}} \neq c/\#(\eta_p(\hat{g}^0) \cap \mathcal{K}(\hat{\alpha}))$ for some $p \in \mathcal{P}(\hat{\alpha}) \cap \eta_k(\hat{g}^0)$ and $t'^{p}_{\tilde{k}} = \hat{t}^p_{\tilde{k}}$ otherwise. Condition (iv) implies that $r_p(\hat{\alpha}, \hat{g}^0, 0) = r_p(\hat{\alpha}, \hat{g}^0[N - p'], 0)$ for $p, p' \in \mathcal{P}(\hat{\alpha})$ such that $pp' \in \hat{G}$ and $\{p, p'\} \cap \bigcup\limits_{k \in \mathcal{K}(\hat{\alpha})} \eta_k(\hat{g}^0) = \{p\}$. Thus, condition (iv) and $p' \notin \bigcup\limits_{k \in \mathcal{K}(\hat{\alpha})} \eta_k(\hat{g}^0)$ imply that a transfer profile $\overline{\tau}$ such that $\overline{t}^{p'}_k > 0$ does not solve problem (1) for all $k \in \mathcal{K}(\hat{\alpha})$. Therefore, we can proceed analogously as before to prove that a transfer profile $\tau'$ such that $t'^{p}_{\tilde{k}} > 0$ for some $p \in \mathcal{P}(\hat{\alpha}) \cap [\eta_{\tilde{k}}(\hat{g}^0)]^c$ and $\hat{t}^p_k \neq c/\#(\eta_p(\hat{g}^0) \cap \mathcal{K}(\hat{\alpha}))$ for some $p \in \mathcal{P}(\hat{\alpha}) \cap \eta_k(\hat{g}^0)$ and $t'^{p}_{\tilde{k}} = \hat{t}^{p}_{\tilde{k}}$ otherwise does not does not solve problem (1) for $\tilde{k} \in \mathcal{K}(\alpha)$. Hence, if $\tau'$ is an equilibrium transfer profile, then $\tau' = \hat{\tau}$.

We now complete the proof by proving Proposition 6 for the case $g^*_T = g^{\varnothing}$. Suppose $g^*_T = g^{\varnothing}$. Denote the ex ante production network of an economy $\hat{\alpha}'$ by $\hat{g}^0 = (N, \hat{G}')$. Let $\hat{\alpha}' \in \hat{\mathcal{A}}$ be such that $\omega(\hat{\alpha}') = \big(\{N - \{k_1, k_2\}\}, \{k_1, k_2\}, \hat{G}'\big)$, where

$$\hat{G}' = \begin{cases} \varnothing & \text{if } \; g^*_T \neq g^0 \\ \{k_1 k_2\} & \text{if } \; g^*_T = g^0; \end{cases}$$

and

1. $r_p(\hat{\alpha}', \hat{g}^{0\prime}, 0) > 0$ for all $p \in \mathcal{P}(\hat{\alpha}')$, and
2. $\Pi_k\left(g^{\mathcal{K}}\left(\hat{\alpha}'\right), Y\right) - \epsilon_k = 0$ for all $k_1, k_2 \in \mathcal{K}(\hat{\alpha}')$

The definitions of $\Pi_k\left(g^{\mathcal{K}(\hat{\alpha})}, Y\right)$ and $r_p(\hat{\alpha}, \hat{g}^0, 0)$ imply that conditions (1) and (2) are independent. Moreover, the definitions of $\Pi_k\left(g^{\mathcal{K}}\left(\hat{\alpha}\right), Y\right)$ and $r_p(\hat{\alpha}, \hat{g}^0, 0)$, and $Y \in \mathbb{R}^{n^2}_+$ and $\epsilon \in \mathbb{R}^n$ imply that there exists $\hat{\alpha}$ such that conditions (1) and (2) are satisfied.

Let $\hat{\tau}'$ be an equilibrium transfer profile. Let $\hat{t}'$ be the implied aggregate transfer by $\hat{\tau}'$. The definition of $T_k$ implies that $\hat{t}'$ must be such that $\hat{t}'^{p}_{k} = 0$ for all $k \in \mathcal{K}(\hat{\alpha}')$ and all $p \in \mathcal{P}(\hat{\alpha}')$. Let $\hat{g}'_T$ be the giving network implied by $\hat{\tau}'$. Then, $\hat{g}'_T = g^{\varnothing}$. By the construction of $\hat{G}'$, $g^0 \neq \hat{g}'^0$, which completes the proof. □

**Proposition 7.** *In any social structure, transfers to all the non isolated poor agents located in the first layer, equal or greater than their subsistence needs, are sufficient to sustain the entire ex ante production network. Formally, pick any $\alpha \in \mathcal{A}$. Then, $H(t, \alpha) = g^0$ if, and only if $t^p \geq r_p(\alpha, g^0, 0)$ for all $p \in \mathcal{S}^1(\alpha)$ such that $\eta_p(g^0) \neq \varnothing$.*

*Proof.* Pick any $\alpha \in \mathcal{A}$. Let $\mathcal{P}' = \{p \in \mathcal{P}(\alpha) : \eta_p(g^0) = \varnothing\}$. Let $t$ be an aggregate transfer vector such that $t^p \geq r_p(\alpha, g^0, 0)$ for all $p \in \mathcal{S}^1(\alpha) \cap [\mathcal{P}']^c$. By the definition of layer and the function $r_p$, $r_p(\alpha, g^0, 0) = \epsilon_p - \Pi_p(g^0, Y) > 0$ for all $p \in \mathcal{S}^1(\alpha)$. Then, by the construction of $t$, $\Pi_p(g^0, Y) - \epsilon_p + t^p \geq 0$ for all $p \in \mathcal{S}^1(\alpha) \cap [\mathcal{P}']^c$. By the definition of layer, $\Pi_l(g^0, Y) - \epsilon_l \geq 0$ for all $l \in \bigcup_{i>1} \mathcal{S}^i(\alpha)$. The definition of $N_s$-subnetwork and the definition of $\mathcal{P}'$ imply that $g^0 = g^0[N - \mathcal{P}']$.

Then, $\Pi_p(g^0[N-\mathcal{P}'],Y)-\epsilon_p \geq 0$ for all $p \in \bigcup_{i>1}\mathcal{S}^i(\alpha)$ and $\Pi_p(g^0[N-\mathcal{P}'],Y)-\epsilon_p+t^p \geq 0$ for all $p \in \mathcal{S}^1(\alpha)\cap[\mathcal{P}']^c$. Moreover, $p \in \mathcal{P}'$ implies $p \in \mathcal{S}^1(\alpha)$. Then, $\mathcal{K}(\alpha)\cup\bigcup_{i>1}\mathcal{S}^i(\alpha)\cup(\mathcal{S}^1(\alpha)\cap[\mathcal{P}']^c) = N-\mathcal{P}'$. Hence, $H(t,\alpha) = g^0[N-\mathcal{P}'] = g^0$.

Suppose $H(t,\alpha) = g^0$. By construction, $\mathcal{P}'$ and the definition of $N_s$-subnetwork imply that $g^0 = g^0[N-\mathcal{P}']$. Then, by the definition of $H(t,\alpha)$ and the definition of layer, $\Pi_p(g^0,Y)-\epsilon_p+t^p \geq 0$ for all $p \in \mathcal{S}^1(\alpha)\cap[\mathcal{P}']^c$. Then, $t^p \geq \epsilon_p - \Pi_p(g^0,Y) = r_p(\alpha,g^0,0)$ for all $p \in \mathcal{S}^1(\alpha)\cap[\mathcal{P}']^c$.

□

# Appendix B: Equilibria of $\Gamma_{k_1,k_2}$

***Result 1.*** *For fixed $\tilde{\alpha} \in \mathcal{A}$ such that $\omega(\tilde{\alpha}) = \big(\{S^1(\tilde{\alpha}) = \{j\}\},\mathcal{K}(\tilde{\alpha}) = \{k_1,k_2\},G^0 = \{k_1j,k_2j\}\big)$, an equilibrium profile $\tau(\tilde{\alpha}) = (t^*_{k_1},t^*_{k_2})$ exists in $\Gamma_{k_1,k_2}$ and is*

*(i) $t^*_k(\tilde{\alpha}) \in \big[\max\{0,r_j(\tilde{\alpha},g^0,0)-\bar{t}_{-k}(\tilde{\alpha})\},\min\{r_j(\tilde{\alpha},g^0,0),\bar{t}_k(\tilde{\alpha})\}\big]$, and*
*$t^*_{-k}(\tilde{\alpha}) = r_j(\tilde{\alpha},g^0,0)-t^*_k(\tilde{\alpha})$, only if $\bar{t}_k(\tilde{\alpha})+\bar{t}_{-k}(\tilde{\alpha}) \geq r_j(\tilde{\alpha},g^0,0)$*

*(ii) $t^*_k(\tilde{\alpha}) = t^*_{-k}(\tilde{\alpha}) = 0$, only if $\bar{t}_k(\tilde{\alpha}) < r_j(\tilde{\alpha},g^0,0)$ and $\bar{t}_{-k}(\tilde{\alpha}) < r_j(\tilde{\alpha},g^0,0)$,*

*for $k,-k \in \{k_1,k_2\}$.*

*Proof.* Fix $\tilde{\alpha} \in \mathcal{A}$ such that $\omega(\tilde{\alpha}) = \big(\{\{j\}\},\{k_1,k_2\},\{k_1j,k_2j\}\big)$. The transfer game is, thus, $\Gamma_{k_1,k_2}$. Let $X = \max\{0,r_j(\tilde{\alpha},g^0,0)-\bar{t}_{-k}(\tilde{\alpha})\}$, $X' = \max\{0,r_j(\tilde{\alpha},g^0,0)-\bar{t}_k(\tilde{\alpha})\}$, $Y = \min\{r_j(\tilde{\alpha},g^0,0),\bar{t}_k(\tilde{\alpha})\}$, and $Y' = \min\{r_j(\tilde{\alpha},g^0,0),\bar{t}_{-k}(\tilde{\alpha})\}$. First, we show that $X \leq Y$ if, and only if $\bar{t}_k(\tilde{\alpha})+\bar{t}_{-k}(\tilde{\alpha}) \geq r_j(\tilde{\alpha},g^0,0)$. Then, we prove that $(t^*_k(\tilde{\alpha}),t^*_{-k}(\tilde{\alpha}))$ such that $t^*_k(\tilde{\alpha}) \in \big[X,Y\big]$ and $t^*_{-k}(\tilde{\alpha}) = r_j(\tilde{\alpha},g^0,0)-t^*_k(\tilde{\alpha})$ are equilibria of $\Gamma_{k_1,k_2}$ if $X \leq Y$. Next, we show that $(t^*_k(\tilde{\alpha}),t^*_{-k}(\tilde{\alpha})) = (0,0)$ is the unique equilibrium of $\Gamma_{k_1,k_2}$ such that $t^*_k(\tilde{\alpha}) \notin [X,Y]$ and $t^*_{-k}(\tilde{\alpha}) \notin [X',Y']$ if $X \leq Y$, $\bar{t}_k(\tilde{\alpha}) < r_j(\tilde{\alpha},g^0,0)$, and $\bar{t}_{-k}(\tilde{\alpha}) < r_j(\tilde{\alpha},g^0,0)$. Then, we prove that $(t^*_k(\tilde{\alpha}),t^*_{-k}(\tilde{\alpha})) = (0,0)$ is the unique equilibrium of $\Gamma_{k_1,k_2}$ if $X > Y$, which completes the proof of (i) and (ii).

We first show that $X \leq Y$ if, and only if $\bar{t}_k(\tilde{\alpha})+\bar{t}_{-k}(\tilde{\alpha}) \geq r_j(\tilde{\alpha},g^0,0)$. There are two cases. First, $X = 0$ if, and only if $\bar{t}_{-k}(\tilde{\alpha}) \geq r_j(\tilde{\alpha},g^0,0)$. By construction $\bar{t}_k(\tilde{\alpha}) \geq 0$. By assumption $r_j(\tilde{\alpha},g^0,0) > 0$. Hence, $Y \geq 0$. Then, $X \leq Y$. Second, by definition, $X = r_j(\tilde{\alpha},g^0,0)-\bar{t}_{-k}(\tilde{\alpha})$ if, and only if $\bar{t}_{-k}(\tilde{\alpha}) \leq r_j(\tilde{\alpha},g^0,0)$, and $Y = r_j(\tilde{\alpha},g^0,0)$ if, and only if $\bar{t}_k(\tilde{\alpha}) \geq r_j(\tilde{\alpha},g^0,0)$.

By construction $\bar{t}_{-k}(\tilde{\alpha}) \geq 0$. Therefore, $r_j(\tilde{\alpha}, g^0, 0) - \bar{t}_{-k}(\tilde{\alpha}) \leq r_j(\tilde{\alpha}, g^0, 0)$ and, thus, $X \leq Y$. Alternatively, $Y = \bar{t}_k(\tilde{\alpha})$ if, and only if $\bar{t}_k(\tilde{\alpha}) \leq r_j(\tilde{\alpha}, g^0, 0)$. Hence, $X \leq Y$ if, and only if $r_j(\tilde{\alpha}, g^0, 0) - \bar{t}_{-k}(\tilde{\alpha}) \leq \bar{t}_k(\tilde{\alpha})$ or, equivalently, $\bar{t}_k(\tilde{\alpha}) + \bar{t}_{-k}(\tilde{\alpha}) \geq r_j(\tilde{\alpha}, g^0, 0)$.

Now we move on to prove that $(t^*_k(\tilde{\alpha}), t^*_{-k}(\tilde{\alpha}))$ such that $t^*_k(\tilde{\alpha}) \in [X, Y]$ and $t^*_{-k}(\tilde{\alpha}) = r_j(\tilde{\alpha}, g^0, 0) - t^*_k(\tilde{\alpha})$ are equilibria of $\Gamma_{k_1,k_2}$ if $X \leq Y$. Let $X \leq Y$and pick $t_k = X$. Then, $X = 0$ if, and only if, $r_j(\tilde{\alpha}, g^0, 0) \leq \bar{t}_{-k}(\tilde{\alpha})$ or $X = r_j(\tilde{\alpha}, g^0, 0) - \bar{t}_{-k}(\tilde{\alpha})$ if, and only if, $r_j(\tilde{\alpha}, g^0, 0) \geq \bar{t}_{-k}(\tilde{\alpha})$. Hence, $X = 0$ implies $r_j(\tilde{\alpha}, g^0, t_k) = r_j(\tilde{\alpha}, g^0, 0)$ and $X = r_j(\tilde{\alpha}, g^0, 0) - \bar{t}_{-k}(\tilde{\alpha})$ implies $r_j(\tilde{\alpha}, g^0, t_k) = \bar{t}_{-k}(\tilde{\alpha})$. Thus, $t_k = X$ implies $r_j(\tilde{\alpha}, g^0, t_k) = Y'$. Analogously, $t_{-k} = X'$ implies $r_j(\tilde{\alpha}, g^0, t_{-k}) = Y$. Now, let $t_k = Y$. Then, $Y = r_j(\tilde{\alpha}, g^0, 0)$ if, and only if, $r_j(\tilde{\alpha}, g^0, 0) - \bar{t}_k \leq 0$ or $Y = \bar{t}_k(\tilde{\alpha})$ if, and only if, $r_j(\tilde{\alpha}, g^0, 0) - \bar{t}_k \geq 0$. Hence, $Y = r_j(\tilde{\alpha}, g^0, 0)$ implies $r_j(\tilde{\alpha}, g^0, t_k) = 0$ and $Y = \bar{t}_k(\tilde{\alpha})$ implies $r_j(\tilde{\alpha}, g^0, t_k) = r_j(\tilde{\alpha}, g^0, 0) - \bar{t}_k(\tilde{\alpha})$. Thus, $t_k = Y$ implies $r_j(\tilde{\alpha}, g^0, t_k) = X'$. Analogously, $t_{-k} = Y'$ implies $r_j(\tilde{\alpha}, g^0, t_{-k}) = X$. To finish this step of the proof, suppose that $X < Y$ and pick $t_k \in (X, Y)$. By definition, $r_j(\tilde{\alpha}, g^0, t_k)$ is continuously decreasing in $t_k \in [X, Y]$. Then, $t_k \in (X, Y)$ implies $r_j(\tilde{\alpha}, g^0, t_k) \in (X', Y')$. Analogously, $t_{-k} \in (X', Y')$ implies $r_j(\tilde{\alpha}, g^0, t_{-k}) \in (X, Y)$. Therefore, $t_k \in [X, Y]$ implies $r_j(\tilde{\alpha}, g^0, t_k) \in [X', Y']$ and $t_{-k} \in [X', Y']$ implies $r_j(\tilde{\alpha}, g^0, t_{-k}) \in [X, Y]$. Moreover, $r_j(\tilde{\alpha}, g^0, t_{-k}) \in [X, Y]$ implies that $r_j(\tilde{\alpha}, g^0, t_{-k}) \leq \bar{t}_k(\tilde{\alpha})$. Then, expression (6) implies that agent $k$'s best response to $t_{-k} \in [X', Y']$ is $t^{br}_k(\tilde{\alpha}, t_{-k}) = r_j(\tilde{\alpha}, g^0, t_{-k})$. By definition of $r_j$, $r_j(\tilde{\alpha}, g^0, t^{br}_k(\tilde{\alpha}, t_{-k})) = t_{-k}$. Analogously $t^{br}_{-k}(\tilde{\alpha}, t_k) = r_j(\tilde{\alpha}, g^0, t_k)$ is the best response to $t_k \in [X, Y]$ and $r_j(\tilde{\alpha}, g^0, t^{br}_{-k}(\tilde{\alpha}, t_k)) = t_k$. Therefore, the continuity of $r_j(\tilde{\alpha}, g^0, t_{-k})$ in $[X, Y]$ and the continuity of $r_j(\tilde{\alpha}, g^0, t_k)$ in $[X', Y']$ implies that for each $t_k \in [X, Y]$ there exist $\tilde{t}^{t_k} = t^{br}_{-k}(\tilde{\alpha}, t_k) \in [X', Y']$ such that $t^{br}_k(\tilde{\alpha}, \tilde{t}^{t_k}) = t_k$. Hence, $(t^*_k(\tilde{\alpha}), t^*_{-k}(\tilde{\alpha}))$ such that $t^*_k(\tilde{\alpha}) \in [X, Y]$ and $t^*_{-k}(\tilde{\alpha}) = r_j(\tilde{\alpha}, g^0, 0) - t^*_k(\tilde{\alpha})$ are equilibria of $\Gamma_{k_1,k_2}$ if $X \leq Y$.

Next, we prove that $(t^*_k(\tilde{\alpha}), t^*_{-k}(\tilde{\alpha})) = (0, 0)$ is the unique equilibrium of $\Gamma_{k_1,k_2}$ such that $t^*_k(\tilde{\alpha}) \notin [X, Y]$ and $t^*_{-k}(\tilde{\alpha}) \notin [X', Y']$ if $X \leq Y$, $\bar{t}_k(\tilde{\alpha}) < r_j(\tilde{\alpha}, g^0, 0)$, and $\bar{t}_{-k}(\tilde{\alpha}) < r_j(\tilde{\alpha}, g^0, 0)$. Pick $t_k \in T_k(\tilde{\alpha})$ such that $t_k > Y$. First, notice that $Y = r_j(\tilde{\alpha}, g^0, 0)$ if, and only if $r_j(\tilde{\alpha}, g^0, 0) \leq \bar{t}_k(\tilde{\alpha})$. Hence, by construction, $t_k > r_j(\tilde{\alpha}, g^0, 0)$. Then, expression (6), $t_k > r_j(\tilde{\alpha}, g^0, 0)$ is not a best response for all $t_{-k} \in T_{-k}(\tilde{\alpha})$. Second, notice that $Y = \bar{t}_k$ if, and only if $r_j(\tilde{\alpha}, g^0, 0) \geq \bar{t}_k$. Hence, by construction, $t_k > \bar{t}_k(\tilde{\alpha})$. Then, expression (6) implies that $t_k > \bar{t}_k(\tilde{\alpha})$ is not a best response for all $t_{-k} \in T_{-k}(\tilde{\alpha})$. Pick now $t_k \in T_k(\tilde{\alpha})$ such that $t_k < X$. First, notice that $X = 0$ if, and only if

$r_j(\tilde{\alpha}, g^0, 0) - \bar{t}_{-k}(\tilde{\alpha}) \leq 0$. Then, by construction $t_k < 0$ which contradicts $t_k \in T_k(\tilde{\alpha})$. Thus, $t_k^*(\tilde{\alpha}) \geq X$ if $X = 0$. Second, notice that $X = r_j(\tilde{\alpha}, g^0, 0) - \bar{t}_{-k}(\tilde{\alpha})$ if, and only if $r_j(\tilde{\alpha}, g^0, 0) - \bar{t}_{-k}(\tilde{\alpha}) \geq 0$. Because we have already showed that $X = 0$ implies $t_k^*(\tilde{\alpha}) \geq X$, it is sufficient to study the case where $\bar{t}_{-k}(\tilde{\alpha}) < r_j(\tilde{\alpha}, g^0, 0)$. By definition, $t_k < X$ implies $t_k < r_j(\tilde{\alpha}, g^0, 0) - \bar{t}_{-k}(\tilde{\alpha})$ or, equivalently, $\bar{t}_{-k}(\tilde{\alpha}) < r_j(\tilde{\alpha}, g^0, 0) - t_k$. Then, $r_j(\tilde{\alpha}, g^0, t_k) > \bar{t}_{-k}(\tilde{\alpha})$. Therefore, expression (6) implies $t_{-k}^{br}(\tilde{\alpha}, t_k) = 0$. Moreover, $t_{-k} = 0$ implies $r_j(\tilde{\alpha}, g^0, t_{-k}) = r_j(\tilde{\alpha}, g^0, 0)$. Thus, there are two cases. Consider first the case $r_j(\tilde{\alpha}, g^0, 0) \leq \bar{t}_k$. Then, by expression (6), $t_k^{br}(\tilde{\alpha}, 0) = r_j(\tilde{\alpha}, g^0, 0)$, which contradicts $t_k < X$. Consider now the case $r_j(\tilde{\alpha}, g^0, 0) > \bar{t}_k(\tilde{\alpha})$, then expression (6), implies $t_k^{br}(\tilde{\alpha}, 0) = 0 < X$. Then, $(t_k^*(\tilde{\alpha}), t_{-k}^*(\tilde{\alpha})) = (0,0)$ is the unique equilibrium of $\Gamma_{k_1,k_2}$ such that $t_k^*(\tilde{\alpha}) \notin [X, Y]$ and $t_{-k}^*(\tilde{\alpha}) \notin [X', Y']$ if $X \leq Y$, $\bar{t}_k(\tilde{\alpha}) < r_j(\tilde{\alpha}, g^0, 0)$, and $\bar{t}_{-k}(\tilde{\alpha}) < r_j(\tilde{\alpha}, g^0, 0)$.

In order to complete the proof of (i) and (ii), we show that $(t_k^*(\tilde{\alpha}), t_{-k}^*(\tilde{\alpha})) = (0,0)$ is the unique equilibrium of $\Gamma_{k_1,k_2}$ if $X > Y$. Let $X > Y$. We have proved that $X \leq Y$ if, and only if $\bar{t}_k(\tilde{\alpha}) + \bar{t}_{-k}(\tilde{\alpha}) \geq r_j(\tilde{\alpha}, g^0, 0)$. Hence, $X > Y$ if, and only if $\bar{t}_k(\tilde{\alpha}) + \bar{t}_{-k}(\tilde{\alpha}) < r_j(\tilde{\alpha}, g^0, 0)$. By construction, $\bar{t}_k(\tilde{\alpha}) \geq 0$ and $\bar{t}_{-k}(\tilde{\alpha}) \geq 0$. Then, $\bar{t}_k(\tilde{\alpha}) < r_j(\tilde{\alpha}, g^0, 0)$ and $\bar{t}_{-k}(\tilde{\alpha}) < r_j(\tilde{\alpha}, g^0, 0)$. Moreover, $\bar{t}_k(\tilde{\alpha}) + \bar{t}_{-k}(\tilde{\alpha}) < r_j(\tilde{\alpha}, g^0, 0)$ implies $r_j(\tilde{\alpha}, g^0, t_k) > \bar{t}_k(\tilde{\alpha}) + \bar{t}_{-k}(\tilde{\alpha}) - t_k$. Pick any $t_k$ such that $t_k \leq \bar{t}_k(\tilde{\alpha})$. Then, $\bar{t}_k(\tilde{\alpha}) - t_k(\tilde{\alpha}) \geq 0$ and, thus, $r_j(\tilde{\alpha}, g^0, t_k) > \bar{t}_{-k}(\tilde{\alpha})$. Hence, expression (6), implies $t_{-k}^{br}(\tilde{\alpha}, t_k) = 0$. Moreover, $t_{-k} = 0$ implies $r_j(\tilde{\alpha}, g^0, t_{-k}) = r_j(\tilde{\alpha}, g^0, 0) > \bar{t}_k(\tilde{\alpha})$. Hence, $t_k^{br}(\tilde{\alpha}, 0) = 0$. Therefore, $(t_k^*(\tilde{\alpha}), t_{-k}^*(\tilde{\alpha})) = (0,0)$ is the unique equilibrium of $\Gamma_{k_1,k_2}$ if $X > Y$.

We have proved that (a) $t_k^*(\tilde{\alpha}) \in \left[\max\{0, r_j(\tilde{\alpha}, g^0, 0) - \bar{t}_{-k}(\tilde{\alpha})\}, \min\{r_j(\tilde{\alpha}, g^0, 0), \bar{t}_k(\tilde{\alpha})\}\right]$ and $t_{-k}^*(\tilde{\alpha}) = r_j(\tilde{\alpha}, g^0, 0) - t_k^*(\tilde{\alpha})$ are equilibria of the transfer game if $\bar{t}_k(\tilde{\alpha}) + \bar{t}_{-k}(\tilde{\alpha}) \geq r_j(\tilde{\alpha}, g^0, 0)$, (b) $t_k^*(\tilde{\alpha}) = t_{-k}^*(\tilde{\alpha}) = 0$ is the unique equilibrium of the transfer game such that $t_k^*(\tilde{\alpha}) \notin [X, Y]$ and $t_{-k}^*(\tilde{\alpha}) \notin [X', Y']$ if $X \leq Y$, $\bar{t}_k(\tilde{\alpha}) < r_j(\tilde{\alpha}, g^0, 0)$ and $\bar{t}_{-k}(\tilde{\alpha}) < r_j(\tilde{\alpha}, g^0, 0)$, and (c) $t_k^* = t_{-k}^* = 0$ is the unique equilibrium of the transfer game if $\bar{t}_k + \bar{t}_{-k} < r_j(\tilde{\alpha}, g^0, 0)$. Then, (i) and (ii) is a direct implication of (a) to (c). $\square$

**Corollary 1.** *Let* $g^1 = (\{k_1, k_2, j\}, \{k_1 j, k_2 j\})$, $g^2 = (\{k_1, k_2, j\}, \{k_1 j\})$, *and* $g^3 = (\{k_1, k_2, j\}, \{k_2 j\})$.

*Then, Result 1 implies that the correspondance of equilibrium giving networks is*

$$\Phi(\tilde{\alpha}) = \begin{cases} \{g^{\varnothing}\} & \text{if } \bar{t}_{k_1} + \bar{t}_{k_2} < r_j(\tilde{\alpha}, g^0, 0) \\ \{g^1, g^{\varnothing}\} & \text{if } \bar{t}_{k_1} + \bar{t}_{k_2} \geq r_j(\tilde{\alpha}, g^0, 0),\ \bar{t}_{k_1} < r_j(\tilde{\alpha}, g^0, 0) \text{ and } \bar{t}_{k_2} < r_j(\tilde{\alpha}, g^0, 0) \\ \{g^1, g^2, g^3\} & \text{if } \bar{t}_{k_1} \geq r_j(\tilde{\alpha}, g^0, 0) \text{ and } \bar{t}_{k_2} \geq r_j(\tilde{\alpha}, g^0, 0) \\ \{g^1, g^2\} & \text{if } \bar{t}_{k_1} + \bar{t}_{k_2} \geq r_j(\tilde{\alpha}, g^0, 0),\ \bar{t}_{k_1} \geq r_j(\tilde{\alpha}, g^0, 0) \text{ and } \bar{t}_{k_2} < r_j(\tilde{\alpha}, g^0, 0) \\ \{g^1, g^3\} & \text{if } \bar{t}_{k_1} + \bar{t}_{k_2} \geq r_j(\tilde{\alpha}, g^0, 0),\ \bar{t}_{k_1} < r_j(\tilde{\alpha}, g^0, 0) \text{ and } \bar{t}_{k_2} \geq r_j(\tilde{\alpha}, g^0, 0) \end{cases}$$

# Appendix C: Technical Lemmas

*Let $\tilde{\Theta} = \bigcup_{m \in \tilde{N}} \{\theta_{ml}\}$ such that $\tilde{N} \subset N$, $l \in N$, $l \notin \tilde{N}$, and $\theta_{ml} \in \Theta_{ml}(g^0)$ for all $m \in \tilde{N}$. Then, $\{l\} \subseteq \bigcap_{\theta \in \tilde{\Theta}} \{N^{\theta}\}$. Let $\bar{g} = \sum_{\theta \in \tilde{\Theta}} \theta$. Let $W(\bar{g}) = \{w \in N^{\bar{g}} : w \in N^{\theta} \cap N^{\theta'}$ for $\theta, \theta' \in \tilde{\Theta}\}$. Then, $l \in W(\bar{g})$.*

**Lemma 2.** *Fix $\alpha \in \mathcal{A}$. Suppose $\tilde{\Theta}$ such that $\#\tilde{N} > 1$. If $\bigcap_{\theta \in \tilde{\Theta}} \{N^{\theta}\} = \{l\}$, then there exists $\theta \in \Theta_{m'm''}(\bar{g})$ such that $l \in N^{\theta}$ and $m' \neq m''$ and $m', m'' \in \tilde{N}$.*

*Proof.* Fix $\alpha \in \mathcal{A}$. Suppose $\tilde{\Theta}$ such that $\#\tilde{N} > 1$. Suppose $\bigcap_{\theta \in \tilde{\Theta}} \{N^{\theta}\} = \{l\}$. The proof is trivial for $W(\bar{g}) = \{l\}$. Now, fix $\#W(\bar{g}) = h > 1$. Pick any $\tilde{\theta} \in \Theta_{m'l}(\bar{g})$ such that $m' \in \tilde{N}$. Let $\tilde{W}(\bar{g}) = N^{\tilde{\theta}} \cap W(\bar{g}) = \{\tilde{w}_1, ..., \tilde{w}_{\tilde{h}}\}$ for $1 \leq \tilde{h} \leq h$, be such $d_{\tilde{w}_1 l}(\tilde{\theta}) > d_{\tilde{w}_2 l}(\tilde{\theta}) > .. > d_{\tilde{w}_{\tilde{h}} l}(\tilde{\theta})$ and $\tilde{w}_{\tilde{h}} = l$. The definition of path implies that $\tilde{\theta} = \tilde{\theta}_{m'\tilde{w}_1} + \tilde{\theta}_{\tilde{w}_1\tilde{w}_2} + ... + \tilde{\theta}_{\tilde{w}_{\tilde{h}-1}l}$. The definition of path also implies that $\tilde{\theta} = \tilde{\theta}_{m'\tilde{w}'} + \tilde{\theta}_{\tilde{w}''l}$ for any $\tilde{w}', \tilde{w}'' \in \tilde{W}(\bar{g})$.

Consider first the case in which $w \in W(\bar{g})$ and $w \neq l$ imply that $w \in \bigcap_{\theta \in \tilde{\Theta}'} \{N^{\theta}\}$ for some $\tilde{\Theta}' \subset \tilde{\Theta}$ such that $\#\tilde{\Theta}' = \#\tilde{\Theta} - 1$. Then, for each $\tilde{w}_i \in \tilde{W}(\bar{g})$ such that $\tilde{w}_i \neq l$, there exists a unique $\theta^i \in \tilde{\Theta}$ such that $\tilde{w}_i \notin N^{\theta^i}$. First, consider the case of $\theta^1 \in \tilde{\Theta}$ which implies $\tilde{w}_1 \notin N^{\theta^1}$. Let $\tilde{W}^1(\bar{g}) = \tilde{W}(\bar{g}) \cap N^{\theta^1}$. The definition of $\tilde{\Theta}$ and $l \in \tilde{W}(\bar{g})$ imply that $\tilde{W}^1(\bar{g}) \neq \varnothing$. Then, there exists $\tilde{w}_{i_1} \in \tilde{W}^1(\bar{g})$ such that $d_{\tilde{w}_1 l}(\tilde{\theta}) > d_{\tilde{w}_{i_1} l}(\tilde{\theta}) > d_{\tilde{w} l}(\tilde{\theta})$ for all $\tilde{w} \in \tilde{W}^1(\bar{g})$ such that $\tilde{w} \neq \tilde{w}_{i_1}$ and $\tilde{w} \neq \tilde{w}_1$. The definition of path implies that $\theta^1 = \theta^1_{m''\tilde{w}_{i_1}} + \theta^1_{\tilde{w}_{i_1} l}$ for $m'' \in \tilde{N}$. Suppose $m'' = m'$. Thus, $m' = \tilde{w}_1$. Then, $\theta^1 \in \tilde{\Theta}$ such that $\tilde{w}_1 \notin N^{\theta^1}$ implies that $m'' \neq m'$, which is a contradiction. Then, $\#\tilde{\Theta}' = \#\tilde{\Theta} - 1$ and $\#\tilde{N} > 1$ imply that $m'' \neq m'$. Then, there exists a unique $\hat{\theta}^1 \in \Theta_{m''\tilde{w}_{i_1}}(\bar{g})$ such that $\{\tilde{w}_1, ..., \tilde{w}_{i_1 - 1}\} \notin N^{\hat{\theta}^1}$, and $d_{\tilde{w}_1 l}(\tilde{\theta}) > ... > d_{\tilde{w}_{i_1} l}(\tilde{\theta})$, and $N^{\hat{\theta}^1} \cap N^{\tilde{\theta}} = \{\tilde{w}_{i_1}\}$, and $2 \leq i_1 \leq \tilde{h}$

for $m'' \neq m'$. The next step of the proof consists in the study of the two family of cases for $i_1$: $i_1 = \tilde{h}$ and $2 \leq i_1 < \tilde{h}$. With the latter step, we build a set of paths $\hat{\Theta} = \{\hat{\theta}^1, ..., \hat{\theta}^q\}$ such that $1 \leq q \leq \tilde{h} - 1$, where each element of $\hat{\Theta}$ has analogous properties to the ones derived for $\hat{\theta}^1$.

Suppose $i_1 = \tilde{h}$. Then, $N^{\hat{\theta}^1} \cap N^{\tilde{\theta}} = \{l\}$. Then, by the definition of path, there exists $\theta \in \Theta_{m'm''}(\overline{g})$ such that $l \in N^{\theta}$ and $m' \neq m''$ and $m', m'' \in \tilde{N}$. Consider now $2 \leq i_1 < \tilde{h}$. Then, $\#\tilde{\Theta}' = \#\tilde{\Theta} - 1$ implies that there exists a unique $\theta^2 \in \tilde{\Theta}$ such that $\tilde{w}_{i_1} \notin N^{\theta^2}$. Moreover, $\tilde{w}_{i_1} \in N^{\theta^1}$ implies that $\theta^2 \neq \theta^1$. Let $\tilde{W}^2(\overline{g}) = \tilde{W}(\overline{g}) \cap N^{\theta^2}$. The definition of $\tilde{\Theta}$ and $l \in \tilde{W}(\overline{g})$ imply that $\tilde{W}^2(\overline{g}) \neq \varnothing$. Suppose $\tilde{w}_{i_1-1} \notin N^{\theta^2}$. Then, there exists $\theta^1$ and $\theta^2$ such that $\theta^1 \neq \theta^2$, and $\tilde{w}_{i_1-1} \notin N^{\theta^1}$ and $\tilde{w}_{i_1-1} \notin N^{\theta^2}$, which contradicts $\#\tilde{\Theta}' = \#\tilde{\Theta} - 1$. Thus, $\tilde{w}_{i_1-1} \in N^{\theta^2}$. Moreover, $\theta^2 \in \tilde{\Theta}$, $i_1 \neq \tilde{h}$, and $d_{\tilde{w}_{\tilde{h}}l}(\tilde{\theta}) < d_{\tilde{w}l}(\tilde{\theta})$ for all $\tilde{w} \in \tilde{W}(\overline{g})$ such that $\tilde{w} \neq w_{\tilde{h}}$ imply that there exists $\tilde{w}_{i_2} \in \tilde{W}^2(\overline{g})$ such that $d_{\tilde{w}_{i_1}l}(\tilde{\theta}) > d_{\tilde{w}_{i_2}l}(\tilde{\theta}) > d_{\tilde{w}l}(\tilde{\theta})$ for all $\tilde{w} \in \tilde{W}^2(\overline{g})$ such that $\tilde{w} \neq \tilde{w}_{i_1}$ and $\tilde{w} \neq \tilde{w}_{i_2}$. Then the definition of path implies that $\theta^2 = \theta^2_{m\tilde{w}_{i_1-1}} + \theta^2_{\tilde{w}_{i_1-1}\tilde{w}_{i_2}} + \theta^2_{\tilde{w}_{i_2}l}$ for some $m \in \tilde{N}$. Then, there exists a unique $\hat{\theta}^2 \in \Theta_{\tilde{w}_{i_1-1}\tilde{w}_{i_2}}(\overline{g})$ such that $\{\tilde{w}_{i_1}, ..., \tilde{w}_{i_2-1}\} \notin N^{\hat{\theta}^2}$, and $d_{\tilde{w}_{i_1-1}l}(\tilde{\theta}) > d_{\tilde{w}_{i_1}l}(\tilde{\theta}) > ... > d_{\tilde{w}_{i_2}l}(\tilde{\theta})$, and $N^{\hat{\theta}^2} \cap N^{\tilde{\theta}} = \{\tilde{w}_{i_1-1}, \tilde{w}_{i_2}\}$, and $3 \leq i_2 \leq \tilde{h}$. We can proceed analogously to show that there exists a unique $\hat{\theta}^d \in \Theta_{\tilde{w}_{i_{d-1}-1}\tilde{w}_{i_d}}(\overline{g})$ such that $\{\tilde{w}_{i_{d-1}}, ..., \tilde{w}_{i_d-1}\} \notin N^{\hat{\theta}^d}$, and $d_{\tilde{w}_{i_{d-1}-1}l}(\tilde{\theta}) > d_{\tilde{w}_{i_{d-1}}l}(\tilde{\theta}) > ... > d_{\tilde{w}_{i_d}l}(\tilde{\theta})$, and $N^{\hat{\theta}^d} \cap N^{\tilde{\theta}} = \{\tilde{w}_{i_{d-1}-1}\tilde{w}_{i_d}\}$, for $d \in \{2, ..., q\}$ such that $w_{i_q} = w_{\tilde{h}}$. Let $\hat{\Theta} = \{\hat{\theta}^1, ..., \hat{\theta}^q\}$ such that $1 \leq q \leq \tilde{h} - 1$.

Let $\overline{\theta} = \tilde{\theta}_{m',\tilde{w}_{i_1-1}} + \sum_{a=1}^{q} \hat{\theta}^a + \sum_{a=1}^{q-2} \tilde{\theta}_{\tilde{w}_{i_a}\tilde{w}_{i_{a+1}-1}} + \tilde{\theta}_{\tilde{w}_{i_{q-1}}l}$. We show now that the definition of $\tilde{\theta}$, and the definition of $\hat{\theta}^a$ and $\hat{\Theta}$ imply $\overline{\theta} \in \Theta_{m'm''}(\overline{g})$ such that $l \in N^{\overline{\theta}}$ and $m' \neq m''$ and $m', m'' \in \tilde{N}$. Let $\tilde{\tilde{\Theta}} = \{\tilde{\theta}_{m',\tilde{w}_{i_1-1}}, \tilde{\theta}_{\tilde{w}_{i_1}\tilde{w}_{i_2-1}}, ..., \tilde{\theta}_{\tilde{w}_{i_{q-2}}\tilde{w}_{i_{q-1}-1}}, \tilde{\theta}_{\tilde{w}_{i_{q-1}}l}\}$ and let $\tilde{\tilde{N}} = \bigcup_{\theta \in \tilde{\tilde{\Theta}}} \{N^{\theta}\}$. Denote by $\tilde{\theta}^i$ the $i$th element of $\tilde{\tilde{\Theta}}$. First, the definition of $\tilde{\theta}$ and $d_{\tilde{w}_1 l}(\tilde{\theta}) > d_{\tilde{w}_2 l}(\tilde{\theta}) > .. > d_{\tilde{w}_{\tilde{h}}l}(\tilde{\theta})$ directly imply that $N^{\tilde{\theta}^i} \cap N^{\tilde{\theta}^j} = \varnothing$ for any $\tilde{\theta}^i, \tilde{\theta}^j \in \tilde{\tilde{\Theta}}$ such that $i \neq j$. Second, the definition of $\hat{\theta}^a$ directly implies that $N^{\hat{\theta}^a} \cap \tilde{\tilde{N}} = \{\tilde{w}_{i_a}\}$ for $a = 1$ and $N^{\hat{\theta}^a} \cap \tilde{\tilde{N}} = \{\tilde{w}_{i_{a-1}-1}, \tilde{w}_{i_a}\}$ for $a \in \{2, ..., q\}$. Finally, we show now that the construction of $\hat{\Theta}$ implies that $N^{\hat{\theta}^i} \cap N^{\hat{\theta}^j} = \varnothing$ for all $\hat{\theta}^i, \hat{\theta}^j \in \hat{\Theta}$ such that $i \neq j$.

Pick any $\hat{\theta}^i, \hat{\theta}^j \in \hat{\Theta}$ such that $i \neq j$. Let $\hat{\theta}^i \in \Theta_{z_i z'_i}(\overline{g})$ such that $d_{z_i l}(\hat{\theta}^i) > d_{z'_i l}(\hat{\theta}^i)$, and $\hat{\theta}^j \in \Theta_{z_j z'_j}(\overline{g})$ such that $d_{z_j l}(\hat{\theta}^j) > d_{z'_j l}(\hat{\theta}^j)$. The definition of $\hat{\Theta}$ and $i \neq j$ imply that $z_i \neq z_{i'} \neq z_j \neq z_{j'}$. Suppose $N^{\hat{\theta}^i} \cap N^{\hat{\theta}^j} \neq \varnothing$. Then, there exists $\overline{w} \in W(\overline{g})$ such that $\hat{\theta}^i = \hat{\theta}^i_{z_i\overline{w}} + \hat{\theta}^i_{\overline{w}z'_i}$, $\hat{\theta}^j = \hat{\theta}^j_{z_j\overline{w}} + \hat{\theta}^j_{\overline{w}z'_j}$, and $\hat{\theta}^i_{z_i\overline{w}} + \hat{\theta}^j_{\overline{w}z'_j} \in \Theta_{z_i z'_j}(\overline{g})$. Without loss of generality suppose $j > i$. Then $N^{\hat{\theta}^1} \cap N^{\tilde{\theta}} = \{\tilde{w}_{i_1}\}$ and $N^{\hat{\theta}^d} \cap N^{\tilde{\theta}} = \{\tilde{w}_{i_{d-1}-1}\tilde{w}_{i_d}\}$ for $d = \{2, ..., q\}$ and the definition of $\tilde{\theta}$ imply that there exists $\theta' \in \tilde{\Theta}$ such that $\theta' = \hat{\theta}^i_{z_i\overline{w}} + \hat{\theta}^j_{\overline{w}z'_j} + \tilde{\theta}_{z'_j l}$ if $i = 1$ or $\theta' = \tilde{\theta}_{mz_i} + \hat{\theta}^i_{z_i\overline{w}} + \hat{\theta}^j_{\overline{w}z'_j} + \tilde{\theta}_{z'_j l}$ for some $m \in \tilde{N}$ if $i > 1$. By the definition of $\hat{\theta}^i$ there exist $w' \in \tilde{W}(\overline{g})$ such that $w' \notin N^{\hat{\theta}^i}$. By the definition of $\hat{\theta}^j$ there exist $w'' \in \tilde{W}(\overline{g})$ such that $w'' \neq w'$ and $w'' \notin N^{\hat{\theta}^j}$. Therefore, $j > i$ implies that $d_{w'l}(\tilde{\theta}) > d_{w''l}(\tilde{\theta})$. Consider the case $i = 1$. By construction, $w'' \notin \hat{\theta}^j$ implies that $w'' \notin N^{\hat{\theta}^j_{\overline{w}z'_j}}$. By the definition of $\hat{\theta}^j$, then $d_{z_jl}(\tilde{\theta}) > d_{w''l}(\tilde{\theta}) > d_{z'_jl}(\tilde{\theta})$. The definition of path implies that $d_{z'_jl}(\tilde{\theta}) \geq d_{bl}(\tilde{\theta})$ for all $b \in N^{\tilde{\theta}_{z'_jl}}$. Then, $w'' \notin N^{\tilde{\theta}_{z'_jl}}$. Moreover, $i = 1$ implies that $N^{\hat{\theta}^i_{z_i\overline{w}}} \cap \tilde{W}(\overline{g}) = \varnothing$. Then, $w'' \in \tilde{W}(\overline{g})$ implies that $w'' \notin N^{\hat{\theta}^i_{z_i\overline{w}}}$. Hence, $w'' \notin N^{\theta'}$. The construction of $\hat{\Theta}$ implies that there exists $\theta^j \in \tilde{\Theta}$ such that $\theta^j = \tilde{\theta}_{mz_j} + \hat{\theta}^j + \tilde{\theta}_{z'_jl}$ for some $m \in \tilde{N}$. By construction, $\theta' \neq \theta^j$. Moreover, the definition of path and $d_{z_jl}(\tilde{\theta}) > d_{w''l}(\tilde{\theta}) > d_{z'_jl}(\tilde{\theta})$ implies that $w'' \notin N^{\tilde{\theta}_{mz_j}} \cup N^{\tilde{\theta}_{z'_jl}}$. Thus, there exists $\theta' \in \tilde{\Theta}$ and $\theta^j \in \tilde{\Theta}$ such that $\theta' \neq \theta^j$ and $w'' \notin N^{\theta'}$ and $w'' \notin N^{\theta^j}$, which contradicts $\#\tilde{\Theta}' = \#\tilde{\Theta} - 1$. To complete this part of the proof, we show that the same conclusion holds for $i > 1$. Suppose that $i > 1$. By the definition of $\hat{\theta}^i$, then $d_{z_il}(\tilde{\theta}) > d_{w'l}(\tilde{\theta}) > d_{z'_il}(\tilde{\theta})$. The definition of path implies that $d_{bl}(\tilde{\theta}) \geq d_{z_il}(\tilde{\theta})$ for all $b \in N^{\tilde{\theta}_{mz_i}}$. Suppose $w'' \in N^{\tilde{\theta}_{mz_i}}$. Then, $d_{w''l}(\tilde{\theta}) \geq d_{z_il}(\tilde{\theta})$ and, thus, $d_{w''l}(\tilde{\theta}) > d_{w'l}(\tilde{\theta})$, which contradicts $j > i$. Thus, $w'' \notin N^{\tilde{\theta}_{mz_i}}$. Moreover, $i > 1$ implies that $N^{\hat{\theta}_{z_i\overline{w}}} \cap \tilde{W}(\overline{g}) = \{z_i\}$. Suppose $w'' = z_i$. Then, $d_{w''l}(\tilde{\theta}) > d_{w'l}(\tilde{\theta})$ which contradicts $j > i$. We have already proved that $w'' \notin N^{\hat{\theta}^j_{\overline{w}z'_j}} \cup N^{\tilde{\theta}_{z'_jl}}$. Hence, $w'' \notin N^{\theta'}$. By construction, $\theta' \neq \theta^j$. Thus, there exists $\theta' \in \tilde{\Theta}$ and $\theta^j \in \tilde{\Theta}$ such that $\theta' \neq \theta^j$ and $w'' \notin N^{\theta'}$ and $w'' \notin N^{\theta^j}$, which contradicts $\#\tilde{\Theta}' = \#\tilde{\Theta} - 1$.

By the definition of path, $N^{\theta_{ii'}} \cap N^{\theta_{i'i''}} = \{i'\}$ implies that $\theta_{ii'} + \theta_{i'i''}$ is a path between $i$ and $i''$ such that $i' \in N^{\theta_{ii'}+\theta_{i'i''}}$. Therefore, the definition of $\tilde{\theta}$, and the definition of $\hat{\theta}^a$ and $\hat{\Theta}$ imply $\overline{\theta} \in \Theta_{m'm''}(\overline{g})$ such that $l \in N^{\overline{\theta}}$ and $m' \neq m''$ and $m', m'' \in \tilde{N}$

To complete the proof, we generalize now the previous result for the case where $\#\tilde{\Theta}' < \#\Theta - 1$ for some $w \in W(\overline{g})$. The construction of $\overline{\theta}$ implies that $\overline{\theta} = \overline{\theta}_{m'\overline{w}_1} + \overline{\theta}_{\overline{w}_1\overline{w}_2} + ... + \overline{\theta}_{\overline{w}_{\tilde{h}-1}\overline{w}_{\tilde{h}}} + \overline{\theta}_{\overline{w}_{\tilde{h}}\overline{w}_{\tilde{h}+1}} + ...\overline{\theta}_{\overline{w}_{\overline{a}}m''}$, where $\overline{a} = \#\left(N^{\overline{\theta}} \cap W(\overline{g})\right)$, $\overline{w}_i \in N^{\overline{\theta}} \cap W(\overline{g})$ for all $1 \leq i \leq \overline{a}$, and $\overline{w}_{\tilde{h}} = l$. Suppose $\overline{g}$ is such that $w_i \in W(\overline{g})$ and $w_i \neq l$ implies that $w_i \in \bigcap_{\theta\in\tilde{\Theta}''(i)} N^{\theta}$ for some $\tilde{\Theta}''(i) \subset \tilde{\Theta}$ such that $2 \leq \#\tilde{\Theta}''(i) < \#\tilde{\Theta} - 1$. Then, $w_i \in \bigcap_{\theta\in\tilde{\Theta}''(i)} N^{\theta}$ for each $w_i \in (\{\overline{w}_1, \overline{w}_2, ..., \overline{w}_{\overline{a}}\} - \{l\})$. Suppose, without loss of generality that $\overline{\theta} \notin \tilde{\Theta}''(\overline{i})$ and $\overline{\theta} \in \tilde{\Theta}''(i)$ for $i \neq \overline{i}$ for $w_i \in (\{\overline{w}_1, \overline{w}_2, ..., \overline{w}_{\overline{a}}\} - \{l\})$. Suppose first $1 < \overline{i} < \overline{a} - 1$. Then, there exists $\overline{\theta}'_{\overline{w}_{\overline{i}-1}\overline{w}_{\overline{i}+1}} \in \Theta_{\overline{w}_{\overline{i}-1}\overline{w}_{\overline{i}+1}}(\overline{g})$ such that $\overline{w}_{\overline{i}} \notin N^{\overline{\theta}'_{\overline{w}_{\overline{i}-1}\overline{w}_{\overline{i}+1}}}$ and $N^{\overline{\theta}'_{\overline{w}_{\overline{i}-1}\overline{w}_{\overline{i}+1}}} \cap N^{\overline{\theta}} = \{\overline{w}_{\overline{i}-1}\overline{w}_{\overline{i}+1}\}$. Then, there exists $\overline{\theta}' = \overline{\theta}_{m'\overline{w}_1} + \overline{\theta}_{\overline{w}_1\overline{w}_2} + ... + \overline{\theta}'_{\overline{w}_{\overline{i}-1}\overline{w}_{\overline{i}+1}} + ... + \overline{\theta}_{\overline{w}_{\overline{a}-1}m''} \in \Theta_{m'm''}(\overline{g})$ such that $l \in N^{\overline{\theta}}$, $m' \neq m''$ and $m', m'' \in \tilde{N}$. Suppose now $\overline{i} = 1$. Then, an analogous analysis implies that there exists

$\overline{\theta}' = \overline{\theta}'_{m'\overline{w}_2} + ... + \overline{\theta}_{\overline{w}_{\overline{a}-1}m''} \in \Theta_{m'm''}(\overline{g})$ such that $l \in N^{\overline{\theta}}$, $m' \neq m''$ and $m', m'' \in \tilde{N}$. Suppose now $\overline{i} = \overline{a} - 1$. Then, there exists $\overline{\theta}' = \overline{\theta}_{m'\overline{w}_1} + ... + \overline{\theta}'_{\overline{w}_{\overline{a}-2}m''} \in \Theta_{m'm''}(\overline{g})$ such that $l \in N^{\overline{\theta}}$, $m' \neq m''$ and $m', m'' \in \tilde{N}$. Let $\overline{\theta}'(\hat{b}) = \overline{\theta}_{m'\hat{w}_1} + \overline{\theta}_{\hat{w}_1\hat{w}_2} + ... + \overline{\theta}_{\hat{w}_{\hat{b}}m''}$ such that $\{\hat{w}_1, \hat{w}_2, ..., \hat{w}_{\hat{b}}\} \subseteq \{\overline{w}_1, \overline{w}_2, ..., \overline{w}_{\overline{a}}\}$. Let $\Omega = \{\overline{\theta}'(\hat{b}) : 1 \leq \hat{b} \leq \overline{a}\}$. Thus, $\overline{g}$ such that $w_i \in \bigcap_{\theta \in \tilde{\Theta}''(i)} \{N^{\theta}\}$ for all $w_i \in W(\overline{g})$ such that $w_i \neq l$ implies that there exists $\overline{\theta}'(\hat{b}) \in \Theta_{m'm''}(\overline{g})$ such that $\overline{\theta}'(\hat{b}) \in \Omega$, $l \in N^{\overline{\theta}'(\hat{b})}$, $m' \neq m''$ and $m', m'' \in \tilde{N}$.

□

**Lemma 3.** *Fix $\alpha \in \mathcal{A}$. If $p', p'' \in N^{DIF(\alpha)}$ such that $\overline{\theta} \in \Theta_{p'p''}(g^0)$, then $p \in N^{\overline{\theta}}$ implies $p \in N^{DIF(\alpha)}$.*

*Proof.* Fix $\alpha \in \mathcal{A}$. Pick any $p', p'' \in N^{DIF(\alpha)}$ such that $\overline{\theta} \in \Theta_{p'p''}(g^0)$. Suppose $N^{\overline{\theta}} \nsubseteq N^{DIF(\alpha)}$. Let $\overline{g} = \overline{\theta} + DIF(\alpha)$. Let $W(\overline{g}) = \{w \in N : w \in N^{\overline{\theta}} \cap N^{DIF(\alpha)}\}$. Then $p', p'' \in W(\overline{g})$. Let $w_i$ be the $i$'th element of the set $W(\overline{g})$. Then, the definition of path implies that $\overline{\theta} = \sum_{i=1}^{\#W(\overline{g})-1} \overline{\theta}_{w_i w_{i+1}}$. By assumption, there exists some $l \in N^{\overline{\theta}}$ such that $l \notin N^{DIF(\alpha)}$. Pick any $\overline{\theta}_{w_i w_{i+1}}$ such that $l \in N^{\overline{\theta}_{w_i w_{i+1}}}$ and $l \notin N^{DIF(\alpha)}$. Fix $\overline{\theta}_{w_i w_{i+1}}$ and $l$. By definition, $w_i, w_{i+1} \in N^{DIF(\alpha)}$. Then, there exists $\theta \in \Theta_{w_i k}(\overline{g})$ such that $N^{\theta} \subseteq N^{DIF(\alpha)}$ for some $k \in \mathcal{K}(\alpha)$. There also exists $\hat{\theta} \in \Theta_{w_i j}(\overline{g})$ such that $N^{\hat{\theta}} \subseteq N^{DIF(\alpha)}$ for some $j \in \mathcal{S}^1(\alpha)$. Analogously there exists $\theta \in \Theta_{w_{i+1}k}(\overline{g})$ such that $N^{\theta} \subseteq N^{DIF(\alpha)}$ for some $k \in \mathcal{K}(\alpha)$. There also exists $\hat{\theta} \in \Theta_{w_{i+1}j}(\overline{g})$ such that $N^{\hat{\theta}} \subseteq N^{DIF(\alpha)}$ for some $j \in \mathcal{S}^1(\alpha)$.

Let $\hat{\Omega}'(i) = \{\theta \in \bigcup_{k \in \mathcal{K}(\alpha)} \Theta_{w_i k}(\overline{g}) : N^{\theta} \subseteq N^{DIF(\alpha)}\}$ and $\hat{\Omega}''(i) = \{\theta \in \bigcup_{j \in \mathcal{S}^1(\alpha)} \Theta_{w_i j}(\overline{g}) : N^{\theta} \subseteq N^{DIF(\alpha)}\}$. The definition of path and the construction of $\overline{\theta}$ imply that $\overline{\theta}_{w_i, w_{i+1}} = \overline{\theta}_{w_i l} + \overline{\theta}_{l w_{i+1}}$ such that (a) $N^{\overline{\theta}_{w_i l}} \cap N^{\overline{\theta}_{w_{i+1} l}} = \{l\}$, (b) $N^{\overline{\theta}_{w_i l}} \cap N^{DIF(\alpha)} = \{w_i\}$, and (c) $N^{\overline{\theta}_{l w_{i+1}}} \cap N^{DIF(\alpha)} = \{w_{i+1}\}$. Then, $\theta \in \hat{\Omega}'(i)$ implies that $\theta + \overline{\theta}_{w_i l} \in \Theta_{lk}(\overline{g})$ for some $k \in \mathcal{K}(\alpha)$. Analogously, $\theta \in \hat{\Omega}''(i)$ implies that $\theta + \overline{\theta}_{w_i l} \in \Theta_{lj}(\overline{g})$ for some $j \in \mathcal{S}^1(\alpha)$. The same argument holds for the case $w_{i+1}$. Thus, $\theta \in \hat{\Omega}'(i+1)$ implies that $\theta + \overline{\theta}_{l w_{i+1}} \in \Theta_{lk}(\overline{g})$ for some $k \in \mathcal{K}(\alpha)$, and $\theta \in \hat{\Omega}''(i+1)$ implies that $\theta + \overline{\theta}_{l w_{i+1}} \in \Theta_{lj}(\overline{g})$ for some $j \in \mathcal{S}^1(\alpha)$.

Let $\tilde{\Omega}(i) = \bigcup_{\theta \in \hat{\Omega}(i)} \{\theta + \overline{\theta}_{w_i l}\}$ such that $\hat{\Omega}(i) = \hat{\Omega}'(i) \cup \hat{\Omega}''(i)$, and let $\tilde{\Omega} = \tilde{\Omega}(i) \cup \tilde{\Omega}(i+1)$. Now we prove that $l \notin N^{DIF(\alpha)}$ implies that there exists $l' \neq l$ such that $l' \in \bigcap_{\theta \in \tilde{\Omega}} N^{\theta}$, which contradicts $w_i, w_{i+1} \in N^{DIF(\alpha)}$. By assumption $l \notin N^{DIF(\alpha)}$. By contradiction suppose $\bigcap_{\theta \in \tilde{\Omega}} N^{\theta} = \{l\}$. By Lemma **??** and the construction of $\tilde{\Omega}$, there exists $\theta \in \Theta_{m'm''}(\overline{g})$ such that $l \in N^{\theta}$ and $m' \neq m''$

and $m', m'' \in \mathcal{K}(\alpha) \cup \mathcal{S}^1(\alpha)$. Without loss of generality, suppose $m' \in \mathcal{K}(\alpha)$ and $m'' \in \mathcal{S}^1(\alpha)$. Then, there exists $\theta \in \Theta_{kj}(\overline{g})$ such that $l \in N^\theta$ for some $k \in \mathcal{K}(\alpha)$ and for some $j \in \mathcal{S}^1(\alpha)$. Thus, $l \in N^{DIF(\alpha)}$, which contradicts $l \notin N^{DIF(\alpha)}$. Suppose $m', m'' \in \mathcal{K}(\alpha)$. Fix $\theta \in \Theta_{m'm''}(\overline{g})$ and denote this path by $\theta_{m'm''}$. The construction of $\tilde{\Omega}$ implies that there exists some $\theta \in \Theta_{lj}(\overline{g})$ for some $j \in \mathcal{S}^1(\alpha)$. Pick $\tilde{\theta} \in \tilde{\Omega}$ such that $\tilde{\theta} \in \Theta_{lj''}(\overline{g})$ for some $j'' \in \mathcal{S}^1(\alpha)$. Suppose $N^{\theta_{m'm''}} \cap N^{\tilde{\theta}} = \{l\}$. Then, the definition of path implies that there exists $\theta \in \Theta_{kj}(\overline{g})$ such that $l \in N^\theta$ for some $k \in \mathcal{K}(\alpha)$ and for some $j \in \mathcal{S}^1(\alpha)$. Thus, $l \in N^{DIF(\alpha)}$, which contradicts $l \notin N^{DIF(\alpha)}$. Now suppose that $N^{\theta_{m'm''}} \cap N^{\tilde{\theta}} = \tilde{W}$ with $\{l\} \subset \tilde{W}$. Let $\tilde{w} \in \tilde{W}$ such that $d_{\tilde{w}j''}(\tilde{\theta}) < d_{aj''}(\tilde{\theta})$ for all $a \in \tilde{W}$ such that $a \neq \tilde{w}$. The definition of path implies that $\tilde{\theta} = \tilde{\theta}_{l\tilde{w}} + \tilde{\theta}_{\tilde{w}j''}$ and $\theta_{m'm''} = \theta_{m'\tilde{w}} + \theta_{\tilde{w}m''}$. Therefore, $d_{\tilde{w}j''}(\tilde{\theta}) < d_{aj''}(\tilde{\theta})$ for all $a \neq \tilde{w}$ and $a \in \tilde{W}$ implies that $N^{\theta_{m'm''}} \cap N^{\tilde{\theta}_{\tilde{w}j}} = \{\tilde{w}\}$. The definition of path also implies that $\theta_{m'm''} = \theta_{m'l} + \theta_{lm''}$. Suppose $\tilde{w} \in N^{\theta_{m'l}}$. Then, there exists $\theta = \tilde{\theta}_{j''\tilde{w}} + \theta_{\tilde{w}m''} \in \Theta_{m''j''}(\overline{g})$ such that $l \in N^\theta$ for $j'' \in \mathcal{S}^1(\alpha)$ and $m'' \in \mathcal{K}(\alpha)$, which implies that $l \in N^{DIF(\alpha)}$. Suppose $\tilde{w} \in N^{\theta_{lm''}}$. Then, there exists $\theta = \tilde{\theta}_{j''\tilde{w}} + \theta_{\tilde{w}m'} \in \Theta_{m'j''}(\overline{g})$ such that $l \in N^\theta$ for some $j'' \in \mathcal{S}^1(\alpha)$ and $m' \in \mathcal{K}(\alpha)$, which implies that $l \in N^{DIF(\alpha)}$. Therefore, $\tilde{w} \in N^{\theta_{m'l}} \cup N^{\theta_{lm''}}$ such that $m', m'' \in \mathcal{K}(\alpha)$ contradicts $l \notin N^{DIF(\alpha)}$. Analogously $m', m'' \in \mathcal{S}^1(\alpha)$ contradicts $l \notin N^{DIF(\alpha)}$. Hence, $\{l\} \subset \bigcap_{\theta \in \tilde{\Omega}} N^\theta$. Therefore, there exists $l' \neq l$ such that $l' \in \bigcap_{\theta \in \tilde{\Omega}} N^\theta$.

Now we show that the existence of the latter $l'$ contradicts $w_i, w_{i+1} \in N^{DIF(\alpha)}$. The construction of $\tilde{\Omega}$ implies that $N^{\overline{\theta}_{lw_i}} \subseteq N^\theta$ for all $\theta \in \tilde{\Omega}(i)$. The construction of $\tilde{\Omega}$ also implies that $N^{\overline{\theta}_{lw_{i+1}}} \subseteq N^\theta$ for all $\theta \in \tilde{\Omega}(i+1)$. Then $l' \in \bigcap_{\theta \in \tilde{\Omega}} N^\theta$ implies that there are four cases: (i) $l' \in N^{\overline{\theta}_{lw_i}}$ and $l' \in N^{\overline{\theta}_{lw_{i+1}}}$, (ii) $l' \in N^{\overline{\theta}_{lw_i}}$ and $l' \notin N^{\overline{\theta}_{lw_{i+1}}}$, (iii) $l' \notin N^{\overline{\theta}_{lw_i}}$ and $l' \in N^{\overline{\theta}_{lw_{i+1}}}$, or (iv) $l' \notin N^{\overline{\theta}_{lw_i}}$ and $l' \notin N^{\overline{\theta}_{lw_{i+1}}}$. First, we assume (i). In this case, $N^{\overline{\theta}_{lw_i}} \cap N^{\overline{\theta}_{lw_{i+1}}} = \{l, l'\}$, which contradicts (a). Second, we assume (ii). By the construction of $\tilde{\Omega}$, $l' \notin N^{\overline{\theta}_{lw_{i+1}}}$ implies that $l' \in (N^\theta - \{w_{i+1}\})$ for some $\theta \in \hat{\Omega}(i+1)$. Then, the definition of $\hat{\Omega}(i+1)$ implies that $l' \in N^{DIF(\alpha)}$ . Thus, $l' \in N^{\overline{\theta}_{lw_i}}$ implies that $N^{\overline{\theta}_{lw_i}} \cap N^{DIF(\alpha)} = \{l', w_i\}$, which contradicts (b). Third, we assume (iii). By the construction of $\tilde{\Omega}$, $l' \notin N^{\overline{\theta}_{lw_i}}$ implies that $l' \in (N^\theta - \{w_i\})$ for some $\theta \in \hat{\Omega}(i)$. Then, the definition of $\hat{\Omega}(i)$ implies that $l' \in N^{DIF(\alpha)}$ . Thus, $l' \in N^{\overline{\theta}_{lw_{i+1}}}$ implies that $N^{\overline{\theta}_{lw_{i+1}}} \cap N^{DIF(\alpha)} = \{l', w_{i+1}\}$, which contradicts (c). Thus, $l' \notin N^{\overline{\theta}_{lw_i}}$ and $l' \notin N^{\overline{\theta}_{lw_{i+1}}}$.

We show now that the latter set of contradictions imply $w_i, w_{i+1} \notin N^{DIF(\alpha)}$. Pick any $\theta', \theta'' \in \hat{\Omega}(i) \cup \hat{\Omega}(i+1)$. Then $l' \notin N^{\overline{\theta}_{lw_i}}$ and $l' \notin N^{\overline{\theta}_{lw_{i+1}}}$ and $l' \in \bigcap_{\theta \in \tilde{\Omega}} N^\theta$ implies that there exists $l' \neq w_i$ and $l' \neq w_{i+1}$ such that $l' \in N^{\theta'} \cap N^{\theta''}$. Hence, there is no path $\theta \in \Theta_{kj}(\overline{g})$ such that

$w_i \in N^{\theta}$ for some $k \in \mathcal{K}(\alpha)$, and no path $\theta \in \Theta_{kj}(\overline{g})$ such that $w_{i+1} \in N^{\theta}$ for some $j \in \mathcal{S}^1(\alpha)$. Therefore, $\alpha \in \mathcal{A}$ such that $DIF(\alpha)$ is a subnetwork of $\overline{g}$ implies that $w_i, w_{i+1} \notin N^{DIF(\alpha)}$, which is by construction a contradiction. $\square$

**Lemma 4.** *Fix $\alpha \in \mathcal{A}$. Let $p', p'' \in \tilde{\mathcal{P}}(\alpha, g^0)$, $p' \neq p''$, $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$, and $\tilde{p} \in \eta_p(g^0)$. Then, $\tilde{p} \notin \hat{\mathcal{P}}(\alpha, g^0, p'')$.*

*Proof.* We prove Lemma 4 in two parts. First, we show that all the nodes in the shortest path to $p'$ are located in a ramification. Then, we use this result to prove that $p', p'' \in \tilde{\mathcal{P}}(\alpha, g^0)$, $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$, $\tilde{p} \in \eta_p(g^0)$ and $\tilde{p} \in \hat{\mathcal{P}}(\alpha, g^0, p'')$ are incompatible conditions.

Fix $\alpha \in \mathcal{A}$. Let $p', p'' \in \tilde{\mathcal{P}}(\alpha, g^0)$, $p' \neq p''$, $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$, and $\tilde{p} \in \eta_p(g^0)$. Suppose $\tilde{p} \in \hat{\mathcal{P}}(\alpha, g^0, p'')$. The definition of $p$ implies that there exists $\theta' \in \Theta_{pp'}(g^0)$ such that $d_{pp'}(\theta') \leq d_{pm}(g^0)$ for all $m \in \tilde{\mathcal{P}}(\alpha, g^0)$. The definition of $\tilde{p}$ implies that there exists $\theta'' \in \Theta_{\tilde{p}p''}(g^0)$ such that $d_{\tilde{p}p''}(\theta'') \leq d_{\tilde{p}m}(g^0)$ for all $m \in \tilde{\mathcal{P}}(\alpha, g^0)$. Fix $\theta'$ and $\theta''$. Let $Q(\theta') = (N^{\theta'} - \{p, p'\}) \cap N^{DIF(\alpha)}$ and

$Q(\theta'') = (N^{\theta''} - \{\tilde{p}, p''\}) \cap N^{DIF(\alpha)}$. We want to prove that $Q(\theta') = \varnothing$ and $Q(\theta'') = \varnothing$. The proof is trivial for $N^{\theta'} - \{p, p'\} = \varnothing$. Suppose $Q(\theta') \neq \varnothing$. Then, there exists $l \in Q(\theta')$ such that $d_{lp}(\theta') \leq d_{ip}(\theta')$ for all $i \in Q(\theta')$ and $d_{lp}(\theta') < d_{ip}(\theta')$ for all $i \in Q(\theta')$ such that $i \neq l$. The definition of $l$ and $p \in N^{R(\alpha)}$ imply that $\eta_l(\theta') \cap N^{R(\alpha)} \neq \varnothing$. By the definition of subnetwork, $\eta_l(g^0) \cap N^{R(\alpha)} \neq \varnothing$. Therefore, $l \in \tilde{\mathcal{P}}(\alpha, g^0)$. By the definition of path, $l \in N^{\theta'}$ implies that $d_{pl}(\theta') < d_{pp'}(\theta')$, which contradicts $d_{pp'}(\theta') \leq d_{pm}(g^0)$ for all $m \in \tilde{\mathcal{P}}(\alpha, g^0)$. Hence, $Q(\theta') = \varnothing$. The proof to show that $Q(\theta'') = \varnothing$ is analogous.

We prove now that $p', p'' \in \tilde{\mathcal{P}}(\alpha, g^0)$, $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$, and $\tilde{p} \in \eta_p(g^0)$ implies $\tilde{p} \notin \hat{\mathcal{P}}(\alpha, g^0, p'')$. The definition of $\eta$ and $\tilde{p} \in \eta_p(g^0)$ imply that $\Theta_{\tilde{p}p'} \neq \varnothing$. Pick any $\hat{\theta} \in \Theta_{\tilde{p}p'}$. Suppose $\{\tilde{p}\} \subset N^{\hat{\theta}} \cap N^{\theta''}$. The definition of path implies that, if $w \in N^{\theta_{ii'}}$ and $w \neq i$, then $d_{wi'}(\theta_{ii'}) < d_{ii'}(\theta_{ii'})$. Then, there exists $z \in N^{\hat{\theta}} \cap N^{\theta''}$ such that $z \neq \tilde{p}$ and $d_{zp''}(\theta'') \leq d_{ip''}(\theta'')$ for all $i \in N^{\hat{\theta}} \cap N^{\theta''}$ and $d_{zp''}(\theta'') < d_{ip''}(\theta'')$ for all $i \in N^{\hat{\theta}} \cap N^{\theta''}$ and $i \neq z$. The definition of path implies that $\theta'' = \theta''_{\tilde{p}z} + \theta''_{zp''}$ such that $N^{\theta''_{\tilde{p}z}} \cap N^{\theta''_{zp''}} = \{z\}$, and $\hat{\theta} = \hat{\theta}_{\tilde{p}z} + \hat{\theta}_{zp'}$ such that $N^{\hat{\theta}_{\tilde{p}z}} \cap N^{\hat{\theta}_{zp'}} = \{z\}$. Suppose there exists $z' \in N^{\theta''_{zp''}} \cap N^{\hat{\theta}_{zp'}}$ such that $z' \neq z$. If $z = p'$, then $N^{\theta''_{zp''}} \cap N^{\hat{\theta}_{zp'}} = \{p'\}$, which contradicts $z' \neq z$ such that $z' \in N^{\theta''_{zp''}} \cap N^{\hat{\theta}_{zp'}}$. If $z = p''$, then $N^{\theta''_{zp''}} \cap N^{\hat{\theta}_{zp'}} = \{p''\}$, which contradicts $z' \neq z$ such that $z' \in N^{\theta''_{zp''}} \cap N^{\hat{\theta}_{zp'}}$. If, $z \neq p'$ and $z \neq p''$, then the definition of path implies that $d_{z'p''}(\theta''_{zp''}) < d_{zp''}(\theta''_{zp''})$. Then, $\theta'' = \theta''_{\tilde{p}z} + \theta''_{zp''}$ implies that $d_{z'p''}(\theta'') < d_{zp''}(\theta'')$, which contradicts $d_{zp''}(\theta'') \leq d_{ip''}(\theta'')$ for all $i \in N^{\hat{\theta}} \cap N^{\theta''}$. Hence, $N^{\theta''_{zp''}} \cap N^{\hat{\theta}_{zp'}} = \{z\}$. Therefore, there exists $\theta \in \Theta_{p'p''}(g^0)$ such that $z \in N^{\theta}$. Then, Lemma 3 and the definitions of $p'$ and $p''$ imply that $z \in N^{DIF(\alpha)}$, which contradicts $Q(\theta'') = \varnothing$. Hence, $N^{\hat{\theta}} \cap N^{\theta''} = \{\tilde{p}\}$. Then, there exists $\theta \in \Theta_{p'p''}(g^0)$ such that $\tilde{p} \in N^{\theta}$. Therefore, Lemma 3 and the definitions of $p'$ and $p''$ imply that $\tilde{p} \in N^{DIF(\alpha)}$, which contradicts $\tilde{p} \in \hat{\mathcal{P}}(\alpha, g^0, p'')$.

$\square$

**Lemma 5.** *Fix $\alpha \in \mathcal{A}$. Suppose $p' \in \tilde{\mathcal{P}}(\alpha, g^0)$. Then, $\Pi_p(g^0[\hat{\mathcal{P}}(\alpha, g^0, p') \cup \{p'\}], Y) - \epsilon_p \geq 0$ for all $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$.*

*Proof.* We prove Lemma 5 in two steps. First, we show that $\eta_p(g^0[\hat{\mathcal{P}}(\alpha, g^0, p') \cup \{p'\}]) = \eta_p(g^0)$ for all $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$. Then, we use the definition of layer to complete the proof.

Fix $\alpha \in \mathcal{A}$. Suppose $p' \in \tilde{\mathcal{P}}(\alpha, g^0)$. The definition of an $N_s$-subnetwork implies that $\eta_p(g^0[\hat{\mathcal{P}}(\alpha, g^0, p') \cup \{p'\}]) \subseteq \eta_p(g^0)$ for all $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$. Pick $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$ and suppose $\eta_p(g^0[\hat{\mathcal{P}}(\alpha, g^0, p') \cup \{p'\}]) \neq \eta_p(g^0)$. Then, there exists $\tilde{p}$ such that $\tilde{p} \in \eta_p(g^0)$ and $\tilde{p} \notin \eta_p(g^0[\hat{\mathcal{P}}(\alpha, g^0, p') \cup \{p'\}])$. Now we show the impossibility of the latter.

Suppose $\tilde{p} \in N^{DIF(\alpha)}$. By the definition of $\hat{\mathcal{P}}$, $p \in N^{R(\alpha)}$. Then, by the definition of $\tilde{\mathcal{P}}$, $\tilde{p} \in \eta_p(g^0)$ implies $\tilde{p} \in \tilde{\mathcal{P}}(\alpha, g^0)$. By construction $\tilde{p} \in \eta_p(g^0)$, which implies that $d_{p\tilde{p}}(g^0) \leq d_{pp'}(g^0)$. Suppose $d_{p\tilde{p}}(g^0) = d_{pp'}(g^0)$. Then, $\tilde{p} \in \eta_p(g^0)$ implies that $p' \in \eta_p(g^0)$. Therefore, $\tilde{p} \neq p'$ implies that there exists $\theta \in \Theta_{p'\tilde{p}}(g^0)$ such that $p \in N^{\theta}$, which by Lemma 3 contradicts $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$. Hence, $d_{p\tilde{p}}(g^0) < d_{pp'}(g^0)$ which, by the definition of $\hat{\mathcal{P}}$, also contradicts $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$. Thus, $\tilde{p} \in N^{R(\alpha)}$. Therefore, $\tilde{p} \in \eta_p(g^0)$ implies that $\tilde{p} \in \hat{\mathcal{P}}(\alpha, g^0, p')$ or $\tilde{p} \in \hat{\mathcal{P}}(\alpha, g^0, p'')$ for some $p'' \in \tilde{\mathcal{P}}(\alpha, g^0)$. By Lemma 4, $\tilde{p} \notin \hat{\mathcal{P}}(\alpha, g^0, p'')$. If $\tilde{p} \in \hat{\mathcal{P}}(\alpha, g^0, p')$, then $\tilde{p} \in \eta_p(g^0[\hat{\mathcal{P}}(\alpha, g^0, p') \cup \{p'\}])$, which is by construction a contradiction. Hence, $\eta_p(g^0[\hat{\mathcal{P}}(\alpha, g^0, p') \cup \{p'\}]) = \eta_p(g^0)$ for all $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$. Thus, the definition of $\Pi_l$ implies $\Pi_p(g^0[\hat{\mathcal{P}}(\alpha, g^0, p') \cup \{p'\}], Y) = \Pi_p(g^0, Y)$ for all $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$.

Now we use the definition of layer to complete the proof. Suppose $p \in \mathcal{S}^1(\alpha)$. The definition of $p$ implies that there exists $\theta' \in \Theta_{pp'}(g^0)$. The definition of $p'$ implies that there exists $\theta'' \in \Theta_{p'k}(g^0)$ for some $k \in \mathcal{K}(\alpha)$. Fix $k \in \mathcal{K}(\alpha)$. Pick any $\theta' \in \Theta_{pp'}(g^0)$ and any $\theta'' \in \Theta_{p'k}(g^0)$. Then, there exists $z \in N^{\theta'} \cap N^{\theta''}$ such that $z \neq k$ and $d_{zk}(\theta'') \leq d_{ik}(\theta'')$ for all $i \in N^{\theta'} \cap N^{\theta''}$ and $d_{zk}(\theta'') < d_{ik}(\theta'')$ for all $i \in N^{\theta'} \cap N^{\theta''}$ and $i \neq z$. The definition of path implies that $\theta' = \theta'_{pz} + \theta'_{zp'}$

such that $N^{\theta'_{pz}} \cap N^{\theta'_{zp'}} = \{z\}$, and $\theta'' = \theta''_{p'z} + \theta''_{zk}$ such that $N^{\theta''_{p'z}} \cap N^{\theta''_{zk}} = \{z\}$. Suppose there exists $z' \in N^{\theta'_{zp'}} \cap N^{\theta''_{zk}}$ such that $z' \neq z$. Then, the definition of path implies that $d_{z'k}(\theta''_{zk}) < d_{zk}(\theta''_{zk})$. Then, $\theta'' = \theta''_{p'z} + \theta''_{zk}$ implies that $d_{z'k}(\theta'') < d_{zk}(\theta'')$, which contradicts $d_{zk}(\theta'') < d_{ik}(\theta'')$ for all $i \in N^{\theta'} \cap N^{\theta''}$ and $i \neq z$. Then, $N^{\theta'_{pz}} \cap N^{\theta''_{zk}} = \{z\}$. Thus, $\theta'_{pz} + \theta''_{zk} \in \Theta_{pk}(g^0)$ for $k \in \mathcal{K}(\alpha)$. Therefore, $p \in \mathcal{S}^1(\alpha)$ implies that $p \in N^{DIF(\alpha)}$, which contradicts $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$. Hence, $p \notin \mathcal{S}^1(\alpha)$. Then, the definition of layer implies $\Pi_p(g^0, Y) - \epsilon_p \geq 0$. Therefore, $p \in \hat{\mathcal{P}}(\alpha, g^0, p')$ implies that $\Pi_p(g^0[\hat{\mathcal{P}}(\alpha, g^0, p') \cup \{p'\}], Y) - \epsilon_p \geq 0$.

□