# Optimal Income Taxation, Public-Goods Provision and Public-Sector Pricing: A Contribution to the Foundations of Public Economics*

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**Abstract**

The paper develops an integrated model of optimal nonlinear income taxation, public-goods provision and pricing in a large economy with private information about labour productivities and public-goods preferences and with binding participation constraints and/or distributive concerns. With binding participation constraints, it is desirable to use nonlinear income taxes as well as admission fees for public-goods finance. With (sufficiently large) inequality aversion, it is desirable to use admission fees as well as income taxes for redistribution. In either case, optimal income taxes and admission fees on excludable public goods satisfy a version of the Mirrlees and Ramsey-Boiteux tax formulae. A renegotiation proofness condition on the consumption side of the allocation is used to write the incentive constraints for the multidimensional hidden characteristics as multiple unidimensional constraints.

*Key Words:* Optimal Income Taxation, Public Goods, Public-Sector Pricing, Multidimensional Mechanism Design, Ramsey-Boiteux Pricing

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# 1 Introduction

This paper develops an integrated model of optimal nonlinear income taxation, public-goods provision and public-sector pricing in a large economy. The analysis shows that the availability of income taxation as a source of government finance alleviates the tension between incentive constraints, financing needs, redistribution concerns, and efficiency in the provision of public goods.

Normative public economics is traditionally divided into three subfields, the theory of public-goods provision, the theory of indirect taxation and public-sector pricing, and the theory of optimal income taxation. The theory of public-goods provision is concerned with the elicitation of preferences for public as opposed to private goods. The Ramsey-Boiteux theory of indirect taxation and public-sector pricing studies the tradeoffs involved in choosing commodity taxes and public-sector prices to finance a public-sector spending requirement when lump sum taxation is unavailable. The theory of optimal income taxation studies equity-efficiency tradeoffs and redistribution when individual productivity levels are unobservable.

These subfields are usually studied separately. Relations between them are little explored. As a prominent exception, Atkinson and Stiglitz (1976) have argued that the Ramsey-Boiteux theory of indirect taxation and public-sector pricing is moot if one allows for direct taxation with nonlinear tax schedules. Christiansen (1981) and Boadway and Keen (1994) have extended the Atkinson-Stiglitz critique to models with public-good provision. However, they assumed that public-goods preferences are common knowledge so that there is no problem of preference revelation.

Conversely, the literature on preference revelation for public-goods provision neglects alternative sources of funds. This literature focuses on the constraints that incentive considerations impose on the relation between expressed public-goods preferences and financial contributions. These constraints are shown to induce a conflict between first-best efficiency, feasibility and individual rationality so that, in some settings, efficient allocations are unattainable, and there is underprovision of any public good that must be financed from voluntary contributions.[1]

Such arguments beg the question why public-goods finance should be limited to payments that people make voluntarily in order to enhance the prospect of public-goods provision. Thus, for public goods that exhibit nonrivalry while being excludable, Schmitz (1997), Norman (2004), and Hellwig

[1] Güth and Hellwig (1986), Rob (1989), Mailath and Postlewaite (1990).

(2007 a) have argued that the detrimental effects of incentive and participation constraints can be diminished if admission fees are used as a source of funds. With nonrivalry in consumption, admission fees induce an inefficiency because people who do not pay the fees will be excluded even though there is no real cost to admitting them. However, the loss from this inefficiency is smaller than the loss from not having the public good provided at all. The logic is familiar from the Ramsey-Boiteux theory of optimal deviations from marginal-cost pricing under a government budget constraint.[2]

For a single excludable public good, Schmitz (1997) and Norman (2004) actually show that, in a large economy with independent private values, Ramsey-Boiteux (or Dupuit) pricing corresponds to an optimal Bayesian mechanism under incentive and participation constraints. For multiple excludable public goods, Hellwig (2007 a) shows that this conclusion is still true if, in addition to feasibility, incentive compatibility, and individual rationality, the incentive mechanism is required to satisfy a condition of *renegotiation proofness.* Under this condition, final allocations of private goods and of admission tickets for public goods must be immune to the possibility that people might engage in unobservable, mutually beneficial side trading.[3]

As usual in the Ramsey-Boiteux approach, with multiple public goods, there is no presumption that any one public good should be self-financing. Some cross-subsidization between public goods is likely to be desirable. Indeed, cross-subsidization is needed to finance non-excludable public goods. The proposition that, in a large economy with independent private values, it is impossible to have national defense expenditures funded on a voluntary basis is moot if revenues from pay TV can be diverted to this purpose!

At this point, however, one must consider the Atkinson-Stiglitz critique of the Ramsey-Boiteux approach. Why use admission fees at all if one can also use direct taxes? Wouldn't a recourse to direct taxation reduce or even eliminate the inefficiencies associated with public-goods finance in a large economy?

The present paper addresses this question. The aim is to integrate the

[2] The link between excludable public goods and the Ramsey-Boiteux theory had already been pointed out by Samuelson (1958). Samuelson insisted that public goods were defined by the property of nonrivalry in consumption and that non-excludability was irrelevant because, in a first-best allocation, the ability to exclude would never be used. However, he did acknowledge the importance of excludability for second-best analysis.

[3] If renegotiation proofness is not imposed, then, in a model with multiple public goods, the Ramsey-Boiteux solution can be dominated by schemes involving mixed bundling, i.e., provision of a combination ticket at a discount relative to the sum of prices for the separate tickets, or even randomized admission schemes; see Fang and Norman (2003/2006), Hellwig (2007 a).

theory of income taxation with the mechanism design approach to the provision of public goods. For public-goods finance, the question is how the availability of income taxation as a source of funds affects the provision and admission pricing of public goods under private information. For distributive policies, the question is how the interaction of income taxation and public-goods provision affect the choice of distortionary instruments for redistribution.

Underlying these questions is a multidimensional mechanism design problem. Atkinson and Stiglitz (1976) had assumed that people differ *only* with respect to their earning abilities; in the public-good provision problem, people also differ with respect to their preferences for public versus private goods.[4] Differences in public-goods preferences are an automatic consequence of the independent-private-values specification for the incentive problem of public-goods provision theory.

As yet, we do not have the tools to solve a mechanism design problem with multidimensional hidden characteristics in full generality.[5] As in Hellwig (2007 a), I therefore impose an additional requirement of *renegotiation proofness*, i.e., the requirement that final allocations of private goods and of admission tickets for excludable public goods must not provide participants with an incentive to engage in (incentive-compatible) side trading. In a large economy, this condition is satisfied if and only if the final allocation of private goods and of admission tickets for public goods is Walrasian. For a quasi-linear specification of public-goods preferences, it follows that final holdings of admission tickets for any public good depend *only* on the values of the preference parameter for that particular public good; in particular, they are independent of earning abilities.

Together with a standard single-crossing property on preferences, this independence property of the consumption side of the allocation yields a complementary independence property for the production side, namely, a person's labour supply and output provision depend *only* on this person's earning ability and are independent of his or her public-goods preferences. Incentive constraints for labour supply and output provision then take the same form as in the standard optimal-income-tax problem with unidimen-

[4] Cremer et al. (2001) have pointed out that Atkinson and Stiglitz rely very heavily on earning abilities being the *only* source of heterogeneity. Allowing for heterogeneity in private-goods endowments as well as earning abilities, they obtain a rationale for redistributive indirect as well as direct taxation.

[5] The problem is exacerbated by the fact that payoffs are not affine in earning abilities. For an account of the difficulties of multidimensional mechanism design with payoff functions that are not affine in the hidden characteristics, see Rochet and Choné (1998).

sional hidden characteristics. The multidimensional mechanism design problem is thus reduced to a problem with multiple unidimensional incentive constraints.

Given this technical result, I study the role of direct taxation in public goods finance. Whereas Hellwig (2007 a) had shown that, in the absence of income taxation, the problem of optimal mechanism design under renegotiation proofness and participation constraints is equivalent to the Ramsey-Boiteux problem of determining optimal admission fees under a government budget constraint, I now show that, if direct taxation is available, some public-goods finance from direct taxaes is *always* desirable.

However, in contrast to Atkinson and Stiglitz (1976), the use of direct taxation as a source of funds does not always make it possible to achieve first-best efficiency when there are no distributive concerns. First-best efficiency requires that public goods be financed by lump-sum payments. Incentive compatibility requires that these payments be the same for everybody. If the payments that are required to finance first-best provision levels for public goods are incompatible with participation constraints for some people, first-best efficiency cannot be achieved.

The question then is what combination of distortionary financing instruments minimizes the total welfare loss from providing the requisite funds for public-goods provision. The optimum is characterized by a combination of Ramsey-Boiteux elasticities rules for admission fees and the Mirrlees formula for marginal income tax rates. In each of these conditions, the efficiency loss from increasing a distortion at the margin is compared to the revenue effects from a higher fee or higher marginal tax rate. At an optimum, the ratio of the two effects is the same for all instruments.

The optimal income tax schedule is nonlinear: it involves a zero marginal tax rate at the top and positive marginal tax rates at all income levels below the top. Given the nonlinearity of the tax schedule, the income tax is *not* redundant and cannot be replaced by an equivalent system of indirect taxes and public-sector prices. Whereas the Ramsey-Boiteux approach has traditionally focussed on the greater flexibility provided by differential commodity taxation where distortionary margins can be adapted to differences in demand elasticities, the analysis here shows that the additional flexibility provided by the nonlinearity of the income tax schedule is also useful. A second-best system of public-goods finance under participation constraints makes active use of both kinds of flexibility.

The second part of the substantive analysis considers the interaction of income taxation and public-goods provision in dealing with distributive concerns. As is well known, *inequality aversion* provides a rationale for distortionary taxation and pricing even when there no participation constraints. Thus, in Atkinson and Stiglitz (1976), as in Mirrlees (1971), inequality aversion gives rise to distortionary income taxation as a tool of redistribution from people with high earning abilities to people with low earning abilities. In Hellwig (2005), differences in the enjoyment that people draw from public goods give rise to distributive concerns; these concerns can justify the use of admission fees as a device for redistribution from people who benefit a lot from the public good to people who do not care for the public good at all.

Whereas Atkinson and Stiglitz (1976) did not allow for any heterogeneity other than differences in earning abilities and Hellwig (2005) did not allow for heterogeneity in earning abilities and for redistribution through income taxation, the analysis here allows for multidimensional heterogeneity and investigates the simultaneous use of income taxes and public-goods admission fees as redistribution devices. With multidimensional heterogeneity, one must worry about correlations between the different parameters. Some correlations will neutralize distributive concerns, others will reinforce them. The paper shows that, if the different productivity and taste parameters are *affiliated*, i.e., independent or positively correlated, then the distributive concerns that are attached to the different parameters do not neutralize each other. The findings from unidimensional analyses are then confirmed or, in the case of positive corrleations, even enhanced.

The paper shows that the traditional alignment of optimal nonlinear income taxation with utilitarian concerns for distribution and of public-goods provision and pricing with government budget constraints and finance is inappropriate. Nonlinear income taxation is important for covering government financing needs, and the pricing of public goods can be important for utilitarian redistribution. Private information about public-goods preferences plays a role in both contexts. The conceptual and mathematical structure of conditions for optimal income tax rates and optimal public-goods admission fees is independent of whether one looks at a problem of public-goods finance under a government budget constraint or at a roblem of utilitarian redistribution.

The appropriate dividing line is not between instruments used for redistribution and instruments used for finance, but between instruments that are vulnerable to arbitrage through side-trading and instruments that are not vulnerable to such arbitrage. We should therefore start thinking in terms a new alignment of linear schemes with renegotiation proofness constraints and nonlinear schemes with an absence of such constraints, of government budget constraints with participation constraints, and of redistribution concerns with inequality aversion in welfare assessments.

In the following, Section 2 lays out the basic model. Section 3 introduces the notion of renegotiation proofness and explains its implications for incentive compatibility and feasibility constraints. Section 4 studies the use of income taxes and admission fees for public-goods finance under participation constraints, Section 5 the use of income taxes and admission fees for redistribution. Proofs are given in the Appendix and in the Supplementary Material.

## 2 A Model with Multiple Public Goods and Endogenous Production

I study a large economy with one private good, $m$ public goods and labour. Some of the public goods are excludable, some are non-excludable. The sets of excludable and non-excludable public goods are denoted as $J^e$ and $J^{ne}$. For each individual $h$ in the economy, an allocation must determine how much of the private good the individual consumes, which public goods he is admitted to and how much labour input he provides. Let $Q_1, ..., Q_m$ be the levels at which public goods $1, ..., m$ are provided. Individual $h$ with taste parameters $\theta_i^h$, $i = 1, ..., m$, obtains the utility

$$c^h + \sum_{i \in J^h} \theta_i^h Q_i - \gamma(y^h, n^h). \tag{2.1}$$

if he has private-good consumption $c^h$, if he is admitted to the enjoyment of public goods $i \in J^h$, and if he works the amount $\gamma(y^h, n^h)$ to provide the output $y^h \geq 0$. In (2.1), no distinction is made between excludable and non-excludable public goods - the individual only cares whether he is actually excluded or not. Trivially though, one must have $J^{ne} \subset J^h$.

The function $\gamma$ in (2.1) is assumed to be twice continuously differentiable, as well as strictly increasing and strictly convex in $y^h$ and nonincreasing in $n^h$. Moreover, $\gamma(0, n^h) = 0$, $\lim_{y^h \downarrow 0} \gamma_y(y^h, n^h) = 0$ for all $n^h$, and $\gamma_{yn}(y^h, n^h) < 0$ for all $y^h$ and $n^h$.

The productivity and taste parameters $n^h$ and $\theta_1^h, ..., \theta_m^h$ differ across agents. The cross-section distribution of the vector $(n^h, \theta_1^h, ..., \theta_m^h)$ over the different agents in the economy is denoted as $F$, with marginal distributions $F^n$ for the productivity parameter $n^h$ and $F^i$ for the taste parameter $\theta_i^h$, $i = 1, ..., m$. The distributions $F, F^n, F^i, i = 1, ..., m$, have strictly positive, continuously differentiable densities $f, f^n$ and $f^i$.

An *allocation* is an array that specifies a nonnegative vector $\mathbf{Q} = (Q_1, ..., Q_m)$ of public-good provision levels and, for each agent $h$, a level $c^h$ of private-good consumption, a set $J^h$ of public goods to which the agent is admitted, and a level $y^h \geq 0$ of output that the agent is required to produce. The analysis will be restricted to allocations that satisfy an *anonymity* condition. Under this condition, $c^h, J^h$, and $y^h$ depend on $h$ *only* through the productivity parameter $n^h$ and the vector $\boldsymbol{\theta}^h = (\theta_1^h, ..., \theta_m^h)$ of taste parameters. Thus, an allocation can be written in the form

$$(\mathbf{Q}, c(\cdot,\cdot), y(\cdot,\cdot), \chi_1(\cdot,\cdot), ..., \chi_m(\cdot,\cdot)), \tag{2.2}$$

where $\mathbf{Q}$ is the vector of public-good provision levels, and, for each $(n, \boldsymbol{\theta}) \in [0,1]^{m+1}$, $c(n, \boldsymbol{\theta})$ and $y(n, \boldsymbol{\theta})$ are the levels of private-good consumption and output provision for an agent with characteristics $(n, \boldsymbol{\theta})$; $\chi_i(n, \boldsymbol{\theta})$ is an indicator variable that takes the value one if the agent is admitted to public good $i$ and the value zero if he is not admitted to public good $i$.

Allocations are assessed according to the welfare functional

$$\int_{[0,1]^{m+1}} W\left(c(n, \boldsymbol{\theta}) + \sum_{i=1}^{m} \chi_i(n, \boldsymbol{\theta})\theta_i Q_i - \gamma(y(n, \boldsymbol{\theta}), n)\right) dF(n, \boldsymbol{\theta}), \tag{2.3}$$

where $W(\cdot)$ is twice continuously differentiable, strictly increasing, and concave. The mechanism design problem will be to choose an allocation that maximizes the welfare functional (2.3) over a set of *admissible* allocations.[6] Admissibility will be defined with reference to conditions of *feasibility, incentive compatibility, renegotiation proofness,* and, in part of the analysis, *individual rationality.*

An allocation is said to be *feasible* if it satisfies

$$\chi_i(n, \boldsymbol{\theta}) \equiv 1 \tag{2.4}$$

for $i \in J^{ne}$ and

$$\int_{[0,1]^{m+1}} c(n, \boldsymbol{\theta}) dF(n, \boldsymbol{\theta}) + \sum_{i=1}^{m} K_i(Q_i) \leq \int_{[0,1]^{m+1}} y(n, \boldsymbol{\theta}) dF(n, \boldsymbol{\theta}). \tag{2.5}$$

These conditions ensure that non-excludable public goods are in fact treated as non-excludable and that aggregate production suffices to provide the resources needed for private consumption and public-goods provision. Public-goods provision costs are assumed to be additively separable. For $i =$

[6] With this formulation, I follow the utilitarian tradition of Mirrlees (1971) and Atkinson (1973). By contrast to the Paretian approach to mechanism design, as embodied in the concept of interim efficiency studied by Ledyard and Palfrey (1999, 2007), the utilitarian approach involves explicit interpersonal utility comparisons and redistribution, as well as concerns about allocative efficiency. The difference is particularly relevant for the analysis of redistributive policies in Section 5 of this paper.

$1, ..., m$, $K_i(Q_i)$ indicates the amount of is the cost of *per capita* private-good consumption that has to be foregone in order to provide the amount $Q_i$ of public good $i$. The function $K_i(\cdot)$ is strictly increasing, strictly convex and twice continuously differentiable, with $K(0) = K_i'(0) = 0$ and $\lim_{Q_i \to \infty} K_i(Q_i) = \infty$.

An allocation is said to be *incentive-compatible* if it satisfies

$$v(n, \boldsymbol{\theta}) \geq c(n', \boldsymbol{\theta}') + \sum_{i=1}^{m} \chi_i(n', \boldsymbol{\theta}')\theta_i Q_i - \gamma(y(n', \boldsymbol{\theta}'), n) \tag{2.6}$$

for all $(n, \boldsymbol{\theta})$ and $(n', \boldsymbol{\theta}')$ in $[0, 1]^{m+1}$, where

$$v(n, \boldsymbol{\theta}) := c(n, \boldsymbol{\theta}) + \sum_{i=1}^{m} \chi_i(n, \boldsymbol{\theta})\theta_i Q_i - \gamma(y(n, \boldsymbol{\theta}), n) \tag{2.7}$$

is the payoff attained by an agent with productivity and preference parameters $(n, \boldsymbol{\theta})$. Each agent $h$ is assumed to have private information about his characteristics $n^h$ and $\boldsymbol{\theta}^h$.[7] Thus, if $(n^h, \boldsymbol{\theta}^h) = (n, \boldsymbol{\theta})$, there is nothing to prevent him from claiming that $(n^h, \boldsymbol{\theta}^h) = (n', \boldsymbol{\theta}')$ in order to obtain the outcome $c(n', \boldsymbol{\theta}')$, $\chi_i(n', \boldsymbol{\theta}'), i = 1, ..., m$, $y(n', \boldsymbol{\theta}')$, instead of the outcome $c(n, \boldsymbol{\theta})$, $\chi_i(n, \boldsymbol{\theta}), i = 1, ..., m$, $y(n, \boldsymbol{\theta})$. Incentive compatibility requires that the agent has nothing to gain from such a claim.[8]

# 3 Renegotiation Proofness

In addition to incentive compatibility, I impose a condition of *renegotiation proofness*. This condition requires that the allocation should not leave any

[7] For the reader who cares about the underlying stochastic specification: We may think of $(n^h, \boldsymbol{\theta}^h)$ as the realization of a random vector $(\tilde{n}^h, \tilde{\boldsymbol{\theta}}^h)$, which is defined on some underlying probability space $(\Omega, \mathcal{F}, P)$ and which has the *ex ante* probability distribution $F$, regardless of $h$. If $h$ is the only person to observe $(n^h, \boldsymbol{\theta}^h)$ and if the random variables $(\tilde{n}^h, \tilde{\boldsymbol{\theta}}^h)$ for different agents are mutually independent, nobody other than $h$ knows anything about $(\tilde{n}^h, \tilde{\boldsymbol{\theta}}^h)$, apart from the fact that $(\tilde{n}^h, \tilde{\boldsymbol{\theta}}^h)$ has the distribution $F$. If, in addition, the random variables $(\tilde{n}^h, \tilde{\boldsymbol{\theta}}^h)$ for the different agents in the continuum economy satisfy a law of large numbers, the cross-section distribution of these random variables is almost surely equal to $F$. As discussed by Judd (1985), such a law of large numbers is consistent with, though not implied by stochastic independence. A large-economy specification with independence in which the law of large numbers holds as a theorem is provided by Al-Najjar (2004).

[8] Along the lines of Mirrlees (1971) and the subsequent literature on optimal income taxation, this treatment of incentive compatibility presumes that $\gamma(y(n, \boldsymbol{\theta}), n)$ is unobservable.

room for mutually beneficial trading of public-goods admission tickets and of the private good among the participants. As in Hammond (1979, 1987), Guesnerie (1995), and Hellwig (2007 a), the mechanism designer is assumed to be unable to prevent people from such side trading. If the allocation the he stipulates leaves room for a Pareto improvement through such side trading, people will make use of this opportunity, and the initial allocation will not actually be the final allocation. Side tading will come to an end when when a renegotiation-proof allocation has been reached. If the mechanism designer is aware of the possibility of renegotiation and if he cares about the allocation that is finally reached rather than the one that is initially stipulated, his choice may be directly expressed in terms of the final renegotiation-proof allocation. Indeed, if he chooses a renegotiation-proof allocation from the beginning, this initial allocation will also be the final allocation.

For a formal treatment, I introduce the concept of a *net-trade allocation* for private-good consumption and public-good admission tickets as an array $(z_c(.,.), z_1(.,.), .., z_m(.,.))$ such that for each $(n, \boldsymbol{\theta})$, $z_c(n, \boldsymbol{\theta})$ and $z_1(n, \boldsymbol{\theta}), .., z_m(n, \boldsymbol{\theta})$ are the net additions to private-good consumption and admission ticket holdings for public goods of a consumer with productivity parameter $n$ and preference parameter vector $\boldsymbol{\theta}$. Given an initial allocation, a net-trade allocation $(z_c(.,.), z_1(.,.), .., z_m(.,.))$ is *feasible* if

$$\chi_i(n, \boldsymbol{\theta}) + z_i(n, \boldsymbol{\theta}) \in \{0, 1\} \tag{3.1}$$

for $i = 1, ..., m$ and all $(n, \boldsymbol{\theta}) \in [0, 1]^{m+1}$, and, moreover,

$$\int_{[0,1]^{m+1}} z_i(n, \boldsymbol{\theta}) dF(n, \boldsymbol{\theta}) = 0 \tag{3.2}$$

for $i = c, 1, ..., m$. Condition (3.1) reflects the fact that admission indicators are binary variables: One either is admitted to a public good or not.[9] Condition (3.2) reflects aggregate resource constraints.

Given an initial allocation, the net-trade allocation $(z_c(.,.), z_1(.,.), .., z_m(.,.))$ is *incentive-compatible* if

$$z_c(n, \boldsymbol{\theta}) + \sum_{i=1}^{m} z_i(n, \boldsymbol{\theta}) \theta_i Q_i \geq z_c(n', \boldsymbol{\theta}') + \sum_{i=1}^{m} z_i(n', \boldsymbol{\theta}') \theta_i Q_i \tag{3.3}$$

[9] In order to eliminate the interdependence of labour input provision and public-good consumption that arises when people with low labour incomes are unable to pay for the admission tickets to the public goods I allow private-good consumption to take negative values.

for all $(n, \boldsymbol{\theta})$ and $(n', \boldsymbol{\theta}')$ in $[0,1]^{m+1}$ for which $\chi_i(n, \boldsymbol{\theta}) + z_i(n', \boldsymbol{\theta}') \in \{0,1\}$ for all $i$. Neither the holdings $(c(n, \boldsymbol{\theta}), \chi_1(n, \boldsymbol{\theta}), ..., \chi_m(n, \boldsymbol{\theta}))$ of private-good consumption and public-goods admission tickets nor the characteristics $(n, \boldsymbol{\theta})$ of a given agent are known by anybody else.[10] If the agent claims that his characteristics are $(n', \boldsymbol{\theta}')$, he obtains the net trade corresponding to these characterictics. Incentive compatibility of the net-trade allocation requires that such a claim must not provide the agent with an improvement over the net trade that is stipulated for him.

An initial allocation is said to be *renegotiation-proof*[11] if, starting from this allocation, there is no feasible and incentive-compatible net-trade allocation which provides a Pareto improvement in the sense that for all $(n, \boldsymbol{\theta}) \in [0,1]^{m+1}$, the utility gain from the net trade $(z_c(n, \boldsymbol{\theta}), z_1(n, \boldsymbol{\theta}), ..., z_m(n, \boldsymbol{\theta}))$ is nonnegative, i.e.

$$z_c(n, \boldsymbol{\theta}) + \sum_{i=1}^{m} z_i(n, \boldsymbol{\theta}) \theta_i Q_i \geq 0, \tag{3.4}$$

and the aggregate utility gain is *strictly* positive, i.e.

$$\int_{[0,1]^{m+1}} [z_c(n, \boldsymbol{\theta}) + \sum_{i=1}^{m} z_i(n, \boldsymbol{\theta}) \theta_i Q_i] \, dF(n, \boldsymbol{\theta}) > 0. \tag{3.5}$$

Renegotiation proofness implies, in particular, that the initial allocation is Pareto efficient. As shown in Hellwig (2007 a), therefore, the fundamental

[10] One might argue that the renegotiation mechanism designer knows the consumer's actual holdings, and therefore the incentive constraints may be loosened. Such loosening of incentive constraints would tend to enhance the scope for renegotiations and make the condition of renegotiation proofness even more restrictive. In the large economy considered here, it does not actually make a difference because the characterization of renegotiation proofness that is given below remains valid. In a finite economy, there would be a difference.

[11] A referee has asked how renegotiation proofness is related to the concept of coalition proofness that was introduced by Laffont and Martimort (1997, 2000). Both concepts rely on the notion that coalition formation in the economy imposes constraints on the design of incentive mechanisms. Whereas Laffont and Martimort have coalitions manipulating the information that serves as an input into the overall incentive mechanism, renegotiation proofness allows for coalitions modifying the outcomes stipulated by the incentive mechanism. Only modifications of outcomes that can be reached by voluntary sidetrading are considered. In the absence of sidetrading of labour, renegotiation proofness does not concern output provision levels. By contrast, the manipulations of information that are considered by Laffont and Martimort can, in principle, affect all outcome dimensions. Moreover, if the coalition in their analysis is a subcoalition of the whole, the implications of its intervention may well be detrimental to people outside that subcoalition, i.e., manipulations are not limited to Pareto improvements.

theorems of welfare economics imply that an allocation is renegotiation-proof if and only if the allocation of private-good consumption and public-goods admission tickets corresponds is a competitive-equilibrium allocation for a system of markets for the private-good and for public-goods admission tickets. This yields a characterization of such allocations in terms of prices.

**Lemma 3.1** *An allocation is renegotiation-proof if and only if there exist prices $p_1, ...p_m$ such that for $i = 1, ..., m$, and almost all $(n, \boldsymbol{\theta}) \in [0,1]^{m+1}$, one has*

$$\chi_i(n, \boldsymbol{\theta}) = 0 \quad \textit{if} \quad \theta_i Q_i < p_i \tag{3.6}$$

*and*

$$\chi_i(n, \boldsymbol{\theta}) = 1 \quad \textit{if} \quad \theta_i Q_i > p_i. \tag{3.7}$$

## 3.1 Implications for Incentive Compatibility

The price characterization of renegotiation-proof allocation can be used to provide an analytically tractable characterization of incentive-compatible allocations that are also renegotiation proof. With renegotiation proofness, the $m+1$-dimensional incentive constraints can actually be decomposed into $m+1$ unidimensional constraints.

**Proposition 3.2** *An allocation is renegotiation-proof and incentive-compatible if and only if there exist prices $p_1, ...p_m$ and functions $\hat{y}(\cdot)$, $\hat{c}(\cdot)$, $\bar{v}(\cdot)$ from $[0,1]$ into $\mathbb{R}_+$ such that the following conditions hold:*

*(a) the expected payoff function $v(\cdot)$ takes the form*

$$v(n, \theta_1, ..., \theta_m) \equiv \bar{v}(n) + \sum_{i=1}^{m} \max(\theta_i Q_i - p_i, 0); \tag{3.8}$$

*(b) for all $(n, \boldsymbol{\theta}) \in [0,1]^{m+1}$, the admission indicators $\chi_i(n, \boldsymbol{\theta})$, $i = 1..., m$, satisfy*

$$\chi_i(n, \boldsymbol{\theta}) = 0 \quad \textit{if} \quad \theta_i Q_i < p_i \quad \textit{and} \quad \chi_i(n, \boldsymbol{\theta}) = 1 \quad \textit{if} \quad \theta_i Q_i > p_i; \tag{3.9}$$

*consumption of the private good satisfies*

$$c(n, \boldsymbol{\theta}) = \bar{v}(n) + \gamma(y(n, \boldsymbol{\theta}), n) - \sum_{i=1}^{m} p_i \chi_i(n, \boldsymbol{\theta}); \tag{3.10}$$

*(c) for all* $\boldsymbol{\theta} \in [0,1]^m$*, the output provision function* $y(\cdot, \boldsymbol{\theta})$ *is nondecreasing and satisfies*

$$y(n, \boldsymbol{\theta}) \equiv \hat{y}(n) \tag{3.11}$$

*for almost all* $n \in [0,1]$*; for any such* $n$*, one also has*

$$c(n, \boldsymbol{\theta}) = \hat{c}(n) - \sum_{i=1}^{m} p_i \chi_i(n, \boldsymbol{\theta}); \tag{3.12}$$

*(d) the functions* $\hat{y}(\cdot)$*,* $\hat{c}(\cdot)$*, and* $\bar{v}(\cdot)$ *satisfy*

$$\bar{v}(n) = \hat{c}(n) - \gamma(\hat{y}(n), n) \tag{3.13}$$

*and*

$$\bar{v}(n) \geq \hat{c}(\hat{n}) - \gamma(\hat{y}(\hat{n}), n) \tag{3.14}$$

*for all* $n$ *and* $\hat{n}$ *in* $[0,1]$.

In interpreting this result, it is useful to go back to Hellwig (2007 a). For a model with *exogenous* production, the main result of that paper showed that an allocation is renegotiation-proof and incentive-compatible if and only if there exist prices $p_1, ...p_m$ such that admission rules satisfy (3.9) and private-good consumption satisfies (3.12) with $\hat{c}(n)$ replaced by the exogenously given endowment $Y$. The prices $p_1, ...p_m$ can be thought of as admission fees. If people can choose whether they want to pay the fee $p_i$ and be admitted to public good $i$ or whether they prefer to forego the enjoyment from public good $i$ and instead consume more of the private good, they will in fact implement the admission rule (3.9), retaining the surplus $\max(\theta_i Q_i - p_i, 0)$ as an information rent.[12]

Proposition 3.2 extends this result to the case of endogenous production. It also shows that, once the implications of renegotiation proofness and incentive compatibility for the consumption side of the economy are taken into account, the implications of incentive compatibility for the production side can be studied without regard for the taste parameters $\theta_1, ..., \theta_m$. Output provision levels are independent of $\theta_1, ..., \theta_m$ (except possibly on a null set). The output $\hat{y}(n)$ entitles an agent to the equivalent of $\hat{c}(n)$ units of the private good, which is also independent of $\theta_1, ..., \theta_m$. He can choose how

[12] Without renegotiation proofness, the information rent might be smaller. In this case, admission rules involving mixed bundling or randomized admissions can be used to extract additional surplus, which can be used, e.g., to provide more of the public-good; see Fang and Norman (2003/2006) or Hellwig (2007 a),

much of this entitlement he wants to spend on admission tickets for excludable public goods and how much on consumption of the private good His consumption of the private good is equal to the difference between the entitlement $\hat{c}(n)$ and the amount he spends on admission tickets for excludable public goods. Because output provision levels and consumption entitlements are independent of $\theta_1, ..., \theta_m$, incentive compatibility for the production side of the economy reduces to the unidimensional condition that

$$\hat{c}(n) - \gamma(\hat{y}(n), n) \geq \hat{c}(\hat{n}) - \gamma(\hat{y}(\hat{n}), n)$$

for all $n$ and $\hat{n}$ in $[0, 1]$.

The fact that $\hat{y}(n)$ is independent of $\theta_1, ..., \theta_m$ is remarkable because the requirement of renegotiation proofness as such does not concern the production side of the economy. In principle, therefore, one could imagine $y(n, \boldsymbol{\theta})$ depending on $\boldsymbol{\theta}$ as well as $n$ even though renegotiation proofness is imposed.

The result rests on two arguments. The first argument combines Lemma 3.1 with incentive compatibility to show that the expected-payoff function $v$ must have the additively separable form (3.8), i.e., that $\theta_1, ..., \theta_m$ affect payoffs only through the surplus $\sum_{i=1}^m \max(\theta_i Q_i - p_i, 0)$

The second argument uses incentive compatibility and the single-crossing condition $\gamma_{ny} < 0$ to show that, for any $\boldsymbol{\theta}$, one must have $y(n, \boldsymbol{\theta}) = y(n, \mathbf{0})$ for almost all $n$. Incentive compatibility requires that, for any $\boldsymbol{\theta}$, one has

$$v(n, \boldsymbol{\theta}) - v(\hat{n}, \boldsymbol{\theta}) \geq -\gamma(y(\hat{n}, \boldsymbol{\theta}), n) + \gamma(y(\hat{n}, \boldsymbol{\theta}), \hat{n}) \tag{3.15}$$

for all $n$ and $\hat{n}$. By standard arguments, this requires that

$$v(n, \boldsymbol{\theta}) - v(\hat{n}, \boldsymbol{\theta}) = -\int_{\hat{n}}^{n} \gamma_n(y(n', \boldsymbol{\theta}), n')dn' \tag{3.16}$$

for all $n$ and $\hat{n}$. By (3.8), it follows that

$$\bar{v}(n) - \bar{v}(\hat{n}) = -\int_{\hat{n}}^{n} \gamma_n(y(n', \boldsymbol{\theta}), n')dn' \tag{3.17}$$

for all $n$ and $\hat{n}$. The function $\bar{v}(\cdot)$ is thus absolutely continuous, and $-\gamma_n(y(\cdot, \boldsymbol{\theta}), \cdot)$ is a version of the Radon-Nikodym derivative of $\bar{v}(\cdot)$, regardless of $\boldsymbol{\theta}$. Therefore, $-\gamma_n(y(n, \boldsymbol{\theta}), n) = -\gamma_n(y(n, \mathbf{0}), n)$ for almost all $n$. Because $\gamma_{ny} < 0$, it follows that $y(n, \boldsymbol{\theta}) = y(n, \mathbf{0})$ for almost all $n$, regardless of $\boldsymbol{\theta}$.

### 3.2 Implications for Feasibility: The Government Budget Constraint

For an allocation that is renegotiation-proof and incentive-compatible, Proposition 3.2 implies that the feasibility conditions (2.4) and (2.5) take the form

$$p_i = 0 \quad \text{for} \quad i \in J^{ne} \tag{3.18}$$

and

$$\sum_{i=1}^{m} K_i(Q_i) \leq \int_0^1 [\hat{y}(n) - \hat{c}(n)]\, dF^n(n) + \sum_{i=1}^{m} p_i(1 - F^i(\hat{\theta}_i)), \tag{3.19}$$

where, for any $i$, $\hat{\theta}_i, p_i, Q_i$ are related by the equation

$$\hat{\theta}_i Q_i = p_i. \tag{3.20}$$

The first term on the right-hand side of (3.19) can be interpreted as aggregate net revenue from direct taxation. By the taxation principle of Hammond (1979) and Guesnerie (1995), the integrand in this term is a function of the output $\hat{y}(n)$ and can be interpreted as an income tax. Formally, there exists a function $T : \mathbb{R}_+ \to \mathbb{R}$ such that the functions $\hat{c}(\cdot)$ and $\hat{y}(\cdot)$ in Proposition 3.2 satisfy

$$\hat{y}(n) - \hat{c}(n) = T(\hat{y}(n)) \tag{3.21}$$

for all $n$, and the constraint (3.19) takes the form

$$\sum_{i=1}^{m} K_i(Q_i) \leq \int_0^1 T(\hat{y}(n))\, dF^n(n) + \sum_{i=1}^{m} p_i(1 - F^i(\hat{\theta}_i)). \tag{3.22}$$

The second term on the right-hand side of (3.19) or (3.22) corresponds to the aggregate revenue from admission fees. Given the fees $p_1, ..., p_m$, for any $i$, there are $(1 - F^i(\hat{\theta}_i))$ participants asking for admission to public good $i$. Aggregate admission fee revenue from public good $i$ is therefore $p_i(1 - F^i(\hat{\theta}_i))$, which is positive if $p_i \in (0, Q_i)$ and zero if $p_i = 0$. Aggregate admission fee revenues from *all* public goods are obtained by summing over all $i = 1, ..., m$.

For a renegotiation-proof and incentive-compatible allocations, the feasibility constraint thus takes the form of a government budget constraint that requires the cost $\sum_i K_i(Q_i)$ to be covered by total revenues from direct taxes and from admission fees for public goods.

# 4 Income Taxes and Public-Goods Finance

## 4.1 The Problem

In this section, I abstract from distributive concerns and assume that the welfare function $W(\cdot)$ is affine, i.e., that $W''(v) = 0$ for all $v$. The welfare functional (2.3) then is ordinally equivalent to the aggregate surplus

$$\int_{[0,1]^{m+1}} [c(n, \boldsymbol{\theta}) + \sum_{i=1}^{m} \chi_i(n, \boldsymbol{\theta})\theta_i Q_i - \gamma(y(n, \boldsymbol{\theta}), n)] \, dF(n, \boldsymbol{\theta}), \tag{4.1}$$

and the cross-section distribution of utility levels is of no concern. I will study allocations that maximize the aggregate surplus (4.1) over the set of admissible allocations where, in addition to feasibility, incentive compatibility, and renegotiation proofness, admissibility also involves a requirement of interim individual rationality. The purpose is to understand the role of direct taxation as a source of *public-goods finance.* Whereas in Hellwig (2007 a), I had assumed that all public-goods finance comes from admission fees, I now take account of the Atkinson-Stiglitz objection that direct taxes should also be allowed for.

To specify participation constraints, I assume that people's outside options are worth zero.[13] An allocation is said to be *individually rational* if the induced payoff function satisfies

$$v(n, \boldsymbol{\theta}) \geq 0 \tag{4.2}$$

for all $(n, \boldsymbol{\theta}) \in [0, 1]^{m+1}$. If the allocation is incentive-compatible and renegotiation proof, $v(\cdot, \cdot)$ is nondecreasing, and (4.2) reduces to the inequality

$$\bar{v}(0) \geq 0, \tag{4.3}$$

where $\bar{v}(0) = \hat{c}(0) - \gamma(\hat{y}(0), 0)$,as specified in Proposition 3.2.

For simplicity, I will refer to an allocation that maximizes the aggregate surplus (4.1) over the set of feasible, incentive-compatible, renegotiation-proof and individually rational allocations as an *optimal admissible allocation.* By Proposition 3.2, renegotiation proofness and incentive compatibility

[13] If one person vetoes the allocation, no alternative scheme is put into place, no production activity occurs, and no public goods are provided. If, instead, outside options allow people to retain surplus from production or to enjoy public goods provided by others, participation constraints would be more restrictive yet. The participation constraint specified in (4.2) is about the weakest that one could imagine in this setting.

imply that for any such allocation, the surplus (4.1) can be written in the form

$$\int_0^1 \bar{v}(n) f^n(n) dn + \sum_{i=1}^m \int_{\hat{\theta}_i}^1 (\theta_i Q_i - p_i)\ dF_i(\theta_i), \tag{4.4}$$

where $\hat{\theta}_i, p_i, Q_i$ are related by (3.20). The public-goods provision levels $Q_1, ..., Q_m$, the admission fees $p_1, ...p_m$, the thresholds $\hat{\theta}_1, ..., \hat{\theta}_m$, and the functions $\hat{y}(\cdot)$, $\hat{c}(\cdot)$, and $\bar{v}(\cdot)$ that are associated with an optimal admissible allocation must therefore be a solution to the problem of maximizing (4.4) subject to the feasibility constraints (3.18) and (3.19), equations (3.20) and (3.13), the incentive compatibility condition (3.14) and the participation constraint (4.3).

In studying this problem, it is convenient to replace $p_i$ and $\hat{c}(n)$ by $\hat{\theta}_i Q_i$ and $\bar{v}(n) + \gamma(\hat{y}(n), n)$, using (3.20) and (3.13). The objective function (4.4) and the feasibility constraint (3.19) then take the form

$$\int_0^1 \bar{v}(n) f^n(n) dn + \sum_{i=1}^m \int_{\hat{\theta}_i}^1 (\theta_i - \hat{\theta}_i) Q_i\ dF_i(\theta_i) \tag{4.5}$$

and

$$\sum_{i=1}^m K_i(Q_i) \leq \int_0^1 [\hat{y}(n) - \gamma(\hat{y}(n), n) - \bar{v}(n)] f^n(n) dn + \sum_{i=1}^m \hat{\theta}_i Q_i (1 - F^i(\hat{\theta}_i)). \tag{4.6}$$

As is well known, the incentive compatibility condition (3.14) can also be replaced by the two conditions that the function $\bar{v}(\cdot)$ be absolutely continuous, with a Radon-Nikodym derivative given as

$$\bar{v}'(n) = -\gamma_n(\hat{y}(n), n) \tag{4.7}$$

for almost all $n$, and that

$$\hat{y}(\cdot) \text{ is a nondecreasing function.} \tag{4.8}$$

Given (4.8), the requirement that $\hat{y}(n)$ be nonnegative for all $n$ reduces to the boundary condition

$$\hat{y}(0) \geq 0. \tag{4.9}$$

One is thus left with the problem of choosing $Q_i$ for $i = 1, ..., m$, $\hat{\theta}_i$ for $i \in J^e$, and the functions $\hat{y}(\cdot)$ and $\bar{v}(\cdot)$ so as to maximize (4.5) subject to (4.6), (4.7), (4.3), (4.8), and (4.9).

### 4.2 Optimality Conditions

The following lemma provides a basis for characterizing optimal admissible allocations.

**Lemma 4.1** *Consider an optimal admissible allocation. Let $p_1, ...p_m$ be the associated admissions prices, and let $\hat{y}(\cdot)$ and $\bar{v}(\cdot)$ be the associated output provision and payoff functions as given by Proposition 3.2. Further, let $\hat{\theta}_1, ..., \hat{\theta}_m$ be such that $\hat{\theta}_i Q_i = p_i$ for all $i$. Then there exists a constant $\lambda \geq 0$ and there exist absolutely continuous functions $\varphi$ and $\psi$ such that the following statements hold:*

**(a)** *for $i = 1, ..., m$,*

$$\int_{\hat{\theta}_i}^{1} (\theta_i - \hat{\theta}_i) dF_i(\theta_i) + \lambda \hat{\theta}_i (1 - F_i(\hat{\theta}_i)) = \lambda K_i'(Q_i) \tag{4.10}$$

**(b)** *for $i \in J^e$,*

$$(\lambda - 1)(1 - F_i(\hat{\theta}_i)) - \lambda \hat{\theta}_i f_i(\hat{\theta}_i) = 0 \tag{4.11}$$

**(c)** *for almost all $n$,*

$$\varphi'(n) = -(1 - \lambda) f^n(n); \tag{4.12}$$

*moreover,*

$$\varphi(0) \leq 0, \quad \varphi(0)\bar{v}(0) = 0, \quad \text{and} \quad \varphi(1) = 0; \tag{4.13}$$

**(d)** *for almost all $n$,*

$$\psi'(n) = -\lambda(1 - \gamma_y(\hat{y}(n), n)) f^n(n) + \varphi(n)\gamma_{ny}(\hat{y}(n), n)); \tag{4.14}$$

*moreover,*

$$\psi(0) \leq 0, \quad \psi(0)\hat{y}(0) = 0, \quad \text{and} \quad \psi(1) = 0; \tag{4.15}$$

**(e)** *for all $n$, $\psi(n) \leq 0$; moreover, $\psi(n) = 0$ if $\hat{y}(\cdot)$ is strictly increasing at $n$.*

Statements (a) and (b) provide the usual first-order conditions for $Q_i$ and $\hat{\theta}_i$. Statements (c) and (d) characterize the costate variables $\varphi$ and $\psi$

that are associated with the state variables $\bar{v}$ and $\hat{y}$. Statement (e) reflects the maximum principle applied to the choice of the "slope" of $\hat{y}(\cdot)$.[14]

It is convenient to eliminate the costate variables $\varphi$ and $\psi$ from these conditions. Given the transversality condition $\varphi(1) = 0$, integration of (4.12) yields

$$\varphi(n) = -(\lambda - 1)(1 - F^n(n)) \tag{4.16}$$

for all $n$. Given the transversality condition $\psi(1) = 0$ and (4.16), integration of (4.14) yields

$$\psi(n) = \int_n^1 \left[\lambda(1 - \gamma_y(\hat{y}(n'), n'))f^n(n') + \gamma_{ny}(\hat{y}(n'), n')(\lambda - 1)(1 - F^n(n'))\right] dn' \tag{4.17}$$

for all $n$. Now, statement (e) of the lemma implies that

$$\int_n^1 \left[\lambda(1 - \gamma_y(\hat{y}(n'), n'))f^n(n') + \gamma_{ny}(\hat{y}(n'), n')(\lambda - 1)(1 - F^n(n'))\right] dn' \leq 0 \tag{4.18}$$

for all $n$. If the inequality in (4.17) is strict, then $\hat{y}(\cdot)$ is constant, i.e., there is bunching, on a neigbourhood of $n$. If $\hat{y}(\cdot)$ is strictly increasing at $n$, the inequality in (4.17) must hold as an equation. If $\hat{y}(\cdot)$ is strictly increasing on a neighbourhood of $n$, the derivative of the integral at $n$ is equal to zero, and one must have

$$\lambda(1 - \gamma_y(\hat{y}(n), n))\ f^n(n) = (-\gamma_{ny}(\hat{y}(n), n))(\lambda - 1)(1 - F^n(n)). \tag{4.19}$$

### 4.3 The Main Result

The qualitative properties of optimal admissible allocations depend on the value of the Lagrange multiplier of the feasibility constraint. By (4.16) and the transversality condition $\varphi(0) \leq 0$, one cannot have $\lambda < 1$. If $\lambda$ were less than one, the value of the Lagrangian could always be increased by adding a constant to the function $\bar{v}(\cdot)$, raising people's private-good consumption at the expense of the government, i.e., the Lagrangian would not have a stationary point.

Whether one has $\lambda = 1$ or $\lambda > 1$ depends on whether first-best levels of public-good provision can be financed by lump-sum taxes without violating

[14] Given that $\hat{y}(\cdot)$ is allowed to be discontinuous, the term "slope" should be taken with a grain of salt. If $\hat{y}(\cdot)$ were known to be piecewise smooth, statement (e) would coincide with the corresponding condition in Ebert (1992), Brunner (1993), or Hellwig (2007 b). The results in Hellwig (2008) imply that statement (e) characterizes the relation between $\hat{y}(\cdot)$ and $\psi(\cdot)$, even if piecewise smoothness is not imposed.

the participation constraint. First-best public-good provision levels are defined by the condition that the marginal cost *per capita* cost of increasing the provision level of any public good $i$ be equal to the cross-section average marginal benefit, i.e., that

$$\int_0^1 \theta_i dF^i(\theta_i) = K_i'(Q_i^*), \quad \text{for} \quad i = 1, ..., m. \tag{4.20}$$

If a lump-sum payment $\sum_i K(Q_i^*)$ by everybody is compatible with individual rationality, the optimal admissible allocation is first-best in the sense that it maximizes (4.1) subject to feasibility only. In this case, one has $\lambda = 1$, and there are no distortions in public-good provision levels, admissions, or output levels. However, if a lump-sum payment $\sum_i K(Q_i^*)$ by everybody is incompatible with individual rationality, one has $\lambda > 1$, and the optimal admissible allocation involves distortions in public-good provision levels, admissions, and output levels. Formally, one obtains:

**Proposition 4.2** *Consider an optimal adissible allocation. Let $p_1, ...p_m$ be the associated admissions prices, and let $\hat{y}(\cdot)$ and $\bar{v}(\cdot)$ be the associated output provision and payoff functions. Further, let $\hat{\theta}_1, ..., \hat{\theta}_m$ be such that $\hat{\theta}_i Q_i = p_i$ for all $i$.*

*(a) If $\sum_i K(Q_i^*) \leq \max_y[y - \gamma(y, 0)]$, then $Q_i = Q_i^*$ and $p_i = 0$ for all $i$. Output provision levels satisfy*

$$\hat{y}(n) = \arg\max_y [y - \gamma(y, n)] \tag{4.21}$$

*for all $n$.*

*(b) If $\sum_i K(Q_i^*) > \max_y[y - \gamma(y, 0)]$, then*

$$0 < Q_i < Q_i^* \tag{4.22}$$

*for all $i$. Moreover, $\hat{\theta}_i > 0$ and*

$$0 < p_i < Q_i \tag{4.23}$$

*for all $i \in J^e$. Output provision levels satisfy*

$$\hat{y}(n) < \arg\max_y [y - \gamma(y, n)] \tag{4.24}$$

*for all $n \in [0, 1)$ and*

$$\lim_{n \uparrow 1} \hat{y}(n) = \arg\max_y [y - \gamma(y, 1)]. \tag{4.25}$$

**Corollary 4.3** ***(a)*** *If* $\sum_i K(Q_i^*) \leq \max_y[y - \gamma(y,0)]$, *the tax schedule* $T(\cdot)$ *that is associated with the optimal admissible allocation in Proposition 4.2 satisfies* $T(y) \equiv \sum_i K(Q_i^*)$. ***(b)*** *If* $\sum_i K(Q_i^*) > \max_y[y - \gamma(y,0)]$, *the tax schedule* $T(\cdot)$ *is strictly increasing on the range of* $\hat{y}(\cdot)$, *i.e.,* $\hat{y}(n) > \hat{y}(\bar{n})$ *implies* $T(\hat{y}(n)) > T(\hat{y}(\bar{n}))$. *If* $n < 1$ *is such that, on a neighbourhood of* $n$, $\hat{y}(\cdot)$ *is continuous and strictly increasing,*[15] *then, on this neighbourhood, the tax schedule* $T(\cdot)$ *is continuously differentiable, with*

$$T'(\hat{y}(n')) = 1 - \gamma_y(\hat{y}(n'), n') \in (0, 1); \tag{4.26}$$

*moreover,*

$$\lim_{y \uparrow \hat{y}(1)} T'(y) = 0 \tag{4.27}$$

If a lump-sum tax $\sum_i K(Q_i^*)$ for everybody is compatible with individual rationality, the Atkinson-Stiglitz critique of the neglect of direct taxation in the Ramsey-Boiteux approach is fully confirmed: In this case, optimal admission fees and marginal income tax rates are zero, and public goods are provided at first-best levels.

If a lump-sum tax $\sum_i K(Q_i^*)$ for everybody is incompatible with individual rationality, the Atkinson-Stiglitz critique is partly confirmed and partly refuted: On the one hand, it is desirable to rely on direct taxation as a source of funds. On the other hand, it is undesirable to rely *only* on direct taxes for public-goods finance. The optimal marginal income tax is positive at all incomes below the maximum. To reduce the distortionary impact of income taxation, admission fees on excludable public goods should also be positive.

## 4.4 Interpretation of the Optimality Conditions

The conditions for an optimal admissible allocation combine elements of the Ramsey-Boiteux and Mirrlees approaches. Except for the value of the Lagrange multiplier $\lambda$, conditions (4.10) and (4.11) are identical to the corresponding conditions for the Ramsey-Boiteux problem studied in Hellwig (2007 a). Equation (4.10) is the relevant version of the Lindahl-Samuelson condition for equality of the social marginal benefits and social marginal costs of an increase in $Q_i$. With $\lambda > 1$, the left-hand side of (4.10), i.e., the

[15] If $|\gamma_{yn}|$ is nonincreasing in $y$, equation (4.19) has no more than one solution for any $n$; by standard arguments, this implies that $\hat{y}(\cdot)$ is continuous. For $\hat{y}(\cdot)$ to be strictly increasing, it is then sufficient that the hazard rate $f^n/(1-F^n)$ and the ratio $\gamma_{yn}/(1-\gamma_y)$ be nondecreasing in $n$.

social marginal benefit of an increasse in $Q_i$, is strictly less than $\lambda$ times $\int_0^1 \theta_i dF^i(\theta_i)$; one therefore has underprovision relative to the first-best level $Q_i^*$.

Condition (4.11) corresponds to the degenerate form that the Ramsey-Boiteux elasticities rule takes when marginal costs of use are equal to zero as they are for a public good exhibiting nonrivalry in consumption. In terms of the admission fee $p_i$, this condition requires that

$$\lambda p_i \; f_i(\frac{p_i}{Q_i}) \; \frac{1}{Q_i} = (\lambda - 1)(1 - F_i(\frac{p_i}{Q_i})). \tag{4.28}$$

The term $1 - F_i(\frac{p_i}{Q_i})$ on the right-hand side of (4.28) indicates the level of aggregate demand for admissions to public good $i$ when the price is $p_i$ and the "quality", i.e. the provision level, is $Q_i$. The term $f^i(\frac{p_i}{Q_i}) \; \frac{1}{Q_i^A}$ on the left-hand side of (4.28) indicates the absolute value of the derivative of this demand with respect to $p_i$. Equation (4.28) requires that the elasticity of demand with respect to $p_i$ be equal to $\frac{\lambda-1}{\lambda}$, regardless of $i$.[16]

As is well known, equation (4.28) calls for a balance between the distributive and the allocative effects of a small change in $p_i$. If $p_i$ is raised by a small amount $\Delta$, there is a distributive effect because people who still ask for admission to the public good now have to pay more; there also is an allocative effect because people who previously were on the margin of indifference now no longer want to pay for admission. The fraction of the population that is affected by the distributive effect is equal to $(1-F^i(\frac{p_i+\Delta}{Q_i}))$, the fraction for which the allocative effect matters is approximately $f_i(\frac{p_i}{Q_i}) \; \frac{1}{Q_i} \; \Delta$. On aggregate, therefore, the price increase redistributes approximately the amount $\Delta \cdot (1 - F_i(\frac{p_i}{Q_i}))$ from the private sector to the public budget; by the allocative effect, it also reduces government revenue by approximately $p_i \cdot f_i(\frac{p_i}{Q_i}) \; \frac{1}{Q_i} \; \Delta$.

[16] In response to a question raised by a referee, I note that, if there was a cost of use, as well as an installation cost for the public goods, the characterization given in Lemma 4.1 and Proposition 4.2 would still be valid *except* that the optimality condition (4.11) would take the form

$$(p_i - \frac{\partial K_i}{\partial U_i}) \; f_i(\frac{p_i}{Q_i}) \; \frac{1}{Q_i} = \frac{\lambda - 1}{\lambda} \; (1 - F_i(\frac{p_i}{Q_i})),$$

where $U_i := \int \pi_i dF_i$ is the aggregate use of public good $i$, and $\frac{\partial K_i}{\partial U_i}$ is the marginal cost of raising $U_i$. A rearrangement of terms now yields the standard form of the inverse-elasticities formula,

$$\frac{p_i - \frac{\partial K}{\partial U_i}}{p_i} = \frac{\lambda - 1}{\lambda} \cdot \frac{1}{\eta_i},$$

where $\eta_i := p_i f_i \frac{1}{Q_i} / (1 - F_i)$ is th eelasticity of demand. If $\frac{\partial K}{\partial U_i} = 0$, this reduces to the condition $\eta_i = \frac{\lambda-1}{\lambda}$, which is discussed in the text.

The right-hand side of (4.28) represents the distributive effect, weighted by the difference $\lambda - 1$ between the marginal values of funds in the public budget and in the private sector; the left-hand side represents the allocative effect, weighted by the marginal value $\lambda$ of additional funds in the public budget. The difference in weights reflects the fact that, for the distributive effect, the government's gain is matched by a private-sector loss; for the allocative effect, the government's loss is not matched by any private-sector gain.

The conditions for optimal marginal income taxes are remarkably similar to the conditions for optimal admission fees. They have the same formal structure as in the Mirrlees approach, but their economic interpretation is in fact the same as the interpretation of the Ramsey-Boiteux conditions for optimal deviations from marginal-cost pricing.

If the function $\hat{y}(\cdot)$ is continuously differentiable and strictly increasing on a neighbourhood of $n$, then, by (4.19) and (4.26), the optimal marginal income tax at $\hat{y}(n)$ satisfies

$$\lambda T'(\hat{y}(n))\, f^n(n) = (\lambda - 1)(-\gamma_{ny})(1 - F^n(n)). \tag{4.29}$$

To interpret this condition, consider the effects of raising the marginal income tax for all $y$ in a small interval $(\hat{y}(n-\delta), \hat{y}(n+\delta))$ by an amount $\Delta$ that is small, even relative to $\delta$. The marginal income tax at other output levels is left unchanged. Like an increase in an admission fee, this increase of the marginal income tax on a small interval has a distributive effect and an allocative effect.

The distributive effect concerns people with earning abilities $n' > n + \delta$. Their income tax payments rise from $T(\hat{y}(n'))$ to $T(\hat{y}(n')) + \Delta(\hat{y}(n+\delta) - \hat{y}(n-\delta))$. If $\hat{y}(\cdot)$ is continuously differentiable, the increase is approximately equal to $\Delta \cdot 2\delta \cdot \frac{d\hat{y}}{dn}(n)$ per person, or $\Delta \cdot 2\delta \cdot \frac{d\hat{y}}{dn}(n) \cdot (1 - F^n(n))$ on aggregate.

The allocative effect concerns people with earning abilities $n' \in (n - \delta, n + \delta)$. If $\Delta$ is small,[17] their output provision levels under the modified tax schedule are given by the equation

$$T'(\hat{y}(n', \Delta)) + \Delta = 1 - \gamma_y(\hat{y}(n', \Delta), n'). \tag{4.30}$$

If the provision function $\hat{y}(\cdot)$ is continuously differentiable, then, by (4.26), the marginal tax schedule $T'(\cdot)$ is also continuously differentiable, and, by

[17] Relative to $n' - (n - \delta)$, of course, $\Delta$ is never small if $n'$ is close to $(n - \delta)$. In this case, the reduction will only go to $\hat{y}(n - \delta)$. At $\hat{y}(n - \delta)$, the new income tax schedule exhibits a kink, with a left-hand derivative equal to $T'(\hat{y}(n - \delta))$ and a right-hand derivative equal to $T'(\hat{y}(n - \delta)) + \Delta$. However, if $\Delta$ is small relative to $\delta$, this complication concerns only a negligible subset of the interval $(n - \delta, n + \delta)$.

the implicit function theorem, so is the function $\hat{y}(\cdot,\cdot)$ in (4.30). The difference $\hat{y}(n',\Delta)-\hat{y}(n')$ is then approximately equal to $\Delta\cdot\frac{\partial\hat{y}}{\partial\Delta}(n',0)$. The impact on the government's tax revenue is approximately $\Delta\cdot T'(\hat{y}(n'))\cdot\frac{\partial\hat{y}}{\partial\Delta}(n',0)$.

From (4.30), the derivatives $\frac{d\hat{y}}{dn}$ and $\frac{\partial\hat{y}}{\partial\Delta}$ are computed as

$$\frac{\partial\hat{y}}{\partial n}=\frac{-\gamma_{yn}}{\gamma_{yy}+T''}\quad\text{and}\quad\frac{\partial\hat{y}}{\partial\Delta}=\frac{-1}{\gamma_{yy}+T''}. \tag{4.31}$$

Thus, if one multiplies the optimality condition (4.29) by $\frac{1}{\gamma_{yy}+T''}$, one can rewrite this equation as

$$\lambda\cdot T'(\hat{y}(n))\cdot\left|\frac{\partial\hat{y}}{\partial\Delta}(n,0)\right|\cdot f^n(n)=(\lambda-1)\cdot\frac{\partial\hat{y}}{\partial n}(n)\cdot(1-F^n(n)). \tag{4.32}$$

The interpretation of this condition is now the same as the interpretation of the Ramsey-Boiteux formula: The right-hand side of (4.32) represents the redistribution effect of a small increase in the marginal tax rate on a small neighbourhood of $\hat{y}(n)$, weighted by the difference between the marginal values of additional funds in the public budget and in the private sector. The left-hand side of (4.32) represents the incentive effect of this increase on people behaviours and thereby on tax revenues, weighted by the marginal value of funds in the public budget.[18]

Equation (4.32) balances the incentive and redistribution effects of a small increase in marginal income tax rates in a neighbourhood of $\hat{y}(n)$, just as equation (4.28) balances the incentive and redistribution effects of a small increase in $p_i$. In each case, the incentive effect is weighted by $\lambda$, and the redistribution effect is weighted by $\lambda-1$. Equations (4.28) and (4.32) *together* imply that, at an optimal admissible allocation, the ratio of the incentive effects to the redistribution effects of small increases in marginal income tax rates and of small increases in admission fees must all be equal to the same ratio $\frac{\lambda-1}{\lambda}$.

From (4.32), one immediately sees that, as in the utilitarian aproach of Mirrlees (1971), the optimal marginal income tax is zero at the top of the

[18] In the context of the Mirrlees model, Roberts (2000) and Saez (2001) provide similar interpretations of the first-order condition for $\hat{y}(\cdot)$ at a point at which there is no bunching. However, they both work with the distribution of income, i.e. $\hat{y}(n)$, rather than the hidden characteristic $n$. Roberts neglects the endogeneity of the income distribution and therefore overlooks the fact that, in (4.32), the terms involving $T''$ cancel out. Saez recognizes the endogeneity of the income distribution and replaces the density of the actual income distribution at any point by the density of the income distribution that would obtain if the actual schedule $T(\cdot)$ were replaced by the linear schedule tangent to $T(\cdot)$ at the point under consideration.

relevant range. Because $(1 - F^n(n))$ is close to zero if $n$ is close to one, the redistribution effect of a small increase in the marginal income tax in a small neighbourhood of $n$ is negligible. In contrast, the optimal marginal income tax is *not* zero at the bottom of the relevant range. Whereas in the utilitarian aproach of Mirrlees (1971), the weight given to the redistribution effect of a small increase in the marginal income tax in a small neighbourhood of $n$ is close to zero if $n$ is close to zero, here, the weight given to the redistribution effect is always $\lambda - 1$.

### 4.5 Comparison to the Simple Ramsey-Boiteux Approach

How does the availability of income taxation as a source of funds affect the provision and admissions pricing of public goods? To answer this question, I compare the optimal admissible allocation in Proposition 4.2 to the optimal allocation in a simple Ramsey-Boiteux approach as studied in Hellwig (2007 a). In the simple Ramsey-Boiteux approach, the government budget constraint for public-goods provision levels and admission fees takes the form

$$\sum_{i=1}^{m} K_i(Q_i) \leq \sum_{i=1}^{m} \hat{\theta}_i(1 - F_i(\hat{\theta}_i)), \tag{4.33}$$

where, for each $i$, $\hat{\theta}_i Q_i = p_i$. This corresponds to a specification of participation constraints so that people are entitled to keep the output they produce, i.e., that

$$v(n, \boldsymbol{\theta}) \geq y(n, \boldsymbol{\theta}) - \gamma(y(n, \boldsymbol{\theta}), n), \tag{4.34}$$

rather than $v(n, \boldsymbol{\theta}) \geq 0$ for all $n$. For an allocation that is renegotiation-proof and incentive-compatible, Proposition 3.2 implies that this stricter participation constraint is equivalent to the requirement that $\hat{c}(n) \geq \hat{y}(n)$ for all $n$ so that the contribution of direct taxation to public-goods finance, i.e., the first term on the right-hand side of (3.19), is at best zero. There may be cross-subsidization between public goods, but there is no subsidization of public-goods provision from direct taxation.

**Proposition 4.4** *Let $Q_1, ..., Q_m$ and $\hat{\theta}_1, ..., \hat{\theta}_m$ be the public-goods provision levels and critical thresholds for admissions that are associated with the optimal admissible allocation in Proposition 4.2. Further, let $\bar{Q}_1, ..., \bar{Q}_m$ and $\bar{\theta}_1, ..., \bar{\theta}_m$ be a solution to the problem of maximizing (4.5) subject to (4.33). If, for $i \in J^e$, $\frac{\theta_i f_i(\theta_i)}{1-F_i(\theta_i)}$ is increasing in $\theta_i$, then*

$$Q_i > \bar{Q}_i \tag{4.35}$$

*for all $i$ and*

$$\hat{\theta}_i < \bar{\theta}_i \tag{4.36}$$

*for $i \in J^e$.*

The availability of income taxation as a source of funds makes it possible to raises public-goods provision levels or to lower the critical thresholds for admission to excludable public goods. If the elasticities $\frac{\theta_i f_i(\theta_i)}{1-F_i(\theta_i)}$, $i \in J^e$, are increasing in $\theta_i$, both of these options are used, i.e., optimal provision levels are higher and optimal admission thresholds are lower when public-goods provision is partly financed by income taxes.

Whether the admission fees $p_i = \hat{\theta}_i Q_i$ are lower than the fees $\bar{p}_i = \bar{\theta}_i \bar{Q}_i$ in the absence of funding from the income tax depends on whether the difference in public-good provision levels outweighs the difference in admission thresholds or not. If the currvature of the cost function $K_i$ is small, the difference between $Q_i$ and $\bar{Q}_i$ is large, and $p_i$ exceeds $\bar{p}_i$. If the curvature of the cost function $K_i$ is large, the difference between $Q_i$ and $\bar{Q}_i$ is small, and $p_i$ is less than $\bar{p}_i$.

In practice, the wisdom of subsidizing excludable public goods from income taxes may be questionable because such subsidies can provide bad incentives for efficiency in the actual provision of such goods. Such incentive effects have been excluded from the analysis here. Developing a tractable model for studying them and for assessing the costs and benefits of subsidization and cross-subsidization of different activities is an important task for future research.

# 5 Utilitarian Redistribution

## 5.1 Optimality Conditions

Turning to the roles of income taxes and admission fees as redistribution devices, in this section, I assume that the welfare function $W(\cdot)$ is strictly concave so that *ceteris paribus* a more even distribution of individual payoffs is preferred to a more dispersed distribution. Following Atkinson (1973), I treat the relative curvature of $W$, $\rho_W(\cdot) = -\frac{W''(\cdot)}{W'(\cdot)}$, as a measure of inequality aversion.

To abstract from the financing roles of income taxes and admission fees, I also assume that there are no partcipation constraints to be met. The problem now is to maximize the functional (2.3) over the set of feasible, incentive-compatible, and renegotiation-proof allocations. A solution to this problem will be referred to as an *optimal utilitarian allocation*. By the same reasoning as in the preceding section, the public-goods provision levels $Q_1, ..., Q_m$, the critical thresholds $\hat{\theta}_1, ..., \hat{\theta}_m$, and the output provision and payoff functions $\hat{y}(\cdot)$ and $\bar{v}(\cdot)$ that are associated with an optimal utilitarian allocation must maximize the functional

$$\int_{[0,1]^{m+1}} W\left(\bar{v}(n) + \sum_{i=1}^{m} \max(\theta_i - \hat{\theta}_i, 0) Q_i\right) dF(n, \boldsymbol{\theta}) \tag{5.1}$$

subject to the feasibility constraints (3.18) and (4.6) and the incentive compatibility condition (3.14). The incentive compatibility condition (3.14) can again be replaced by conditions (4.7) and (4.8). The problem then is to choose $Q_1, ..., Q_m, \hat{\theta}_1, ..., \hat{\theta}_m, \hat{y}(\cdot)$, and $\bar{v}(\cdot)$ so as to maximize (5.1) subject to (4.6), (4.7), (4.8), and (4.9).

The following lemma provides the analogue of Lemma 4.1 for this optimization problem. In this lemma, the expression $V(\bar{v}, n; Q_1, ..., Q_m; \hat{\theta}_1, ..., \hat{\theta}_m)$ refers to the conditional expectation of $W\left(\bar{v} + \sum_{i=1}^{m} \max(\theta_i - \hat{\theta}_i, 0) Q_i\right)$ given $n$; the partial derivative $V_v$ of $V$ with respect to the first argument is equal to the conditional expectation of the derivative $W'\left(\bar{v} + \sum_{i=1}^{m} \max(\theta_i - \hat{\theta}_i, 0) Q_i\right)$ given $n$.

**Lemma 5.1** *Consider an an optimal utilitarian allocation. Let $p_1, ...p_m$ be the associated admissions prices, and let $\hat{y}(\cdot)$ and $\bar{v}(\cdot)$ be the associated output provision and payoff functions. Further, let $\hat{\theta}_1, ..., \hat{\theta}_m$ be such that $\hat{\theta}_i Q_i = p_i$ for all $i$. Then there exists a constant $\lambda \geq 0$ and there exist absolutely continuous functions $\varphi$ and $\psi$ such that the following statements hold:*

**(a)** *for $i = 1, ..., m$,*

$$\int_{[0,1]^{m+1}} W'\left(\bar{v}(n) + \sum_{j=1}^{m} \max(\theta_j - \hat{\theta}_j, 0) Q_j\right) \max(\theta_i - \hat{\theta}_i, 0) dF(n, \boldsymbol{\theta}) \\ + \lambda \hat{\theta}_i (1 - F_i(\hat{\theta}_i)) = \lambda K_i'(Q_i) \tag{5.2}$$

**(b)** *for $i \in J^e$,*

$$\lambda(1 - F_i(\hat{\theta}_i) - \hat{\theta}_i f_i(\hat{\theta}_i)) \\ = \int_{\hat{\theta}_i}^{1} \int_{[0,1]^m} W'\left(\bar{v}(n) + \sum_{j=1}^{m} \max(\theta_j - \hat{\theta}_j, 0) Q_j\right) dF(n, \boldsymbol{\theta}) \tag{5.3}$$

**(c)** *for almost all n,*

$$\varphi'(n) = -(V_v - \lambda) f^n(n); \tag{5.4}$$

*moreover,*

$$\varphi(0) = \varphi(1) = 0; \tag{5.5}$$

**(d)** *for almost all n,*

$$\psi'(n) = -\lambda(1 - \gamma_y(\hat{y}(n), n)) f^n(n) + \varphi(n)\gamma_{ny}(\hat{y}(n), n)); \tag{5.6}$$

*moreover,*

$$\psi(0) \leq 0, \quad \psi(0)\hat{y}(0) = 0, \quad \text{and} \quad \psi(1) = 0; \tag{5.7}$$

**(e)** *for all $n$, $\psi(n) \leq 0$; moreover, $\psi(n) = 0$ if $\hat{y}(\cdot)$ is strictly increasing at $n$.*

The conditions in this lemma and in Lemma 4.1 differ in two respects. First, in (5.2) - (5.4), the slope $W'$ of the welfare function $W$ comes in explicitly. In the corresponding conditions (4.10) - (4.12) in Lemma 4.1, $W'$ does not appear explicitly because the maximand there corresponds to the specification $W' \equiv 1$.

Second, because there is no participation constraint, the transversality condition for $\bar{v}(0)$ takes the form $\varphi(0) = 0$. In combination with (5.4) and the transversality condition for $\varphi(1)$, this gives rise to the equation

$$\lambda = \int_{[0,1]^{m+1}} W' \left( \bar{v}(n) + \sum_{j=1}^{m} \max(\theta_j - \hat{\theta}_j, 0) Q_j \right) dF(n, \boldsymbol{\theta}) \tag{5.8}$$

for the Lagrange multiplier of the feasibility or government budget constraint.

As before, it is convenient to compute $\varphi(n)$ and $\psi(n)$ by integrating $\varphi'$ and $\psi'$ and using the transversality conditions. This yields

$$\varphi(n) = \int_n^1 (V_v - \lambda) dF^n(n') \tag{5.9}$$

and

$$\psi(n) = \int_0^1 [\lambda(1 - \gamma_y(\hat{y}(n'), n')) f^n(n') - \varphi(n')\gamma_{ny}(\hat{y}(n'), n'))] dn'. \tag{5.10}$$

Now, statement (e) of the lemma implies that

$$\int_n^1 \left[\lambda(1-\gamma_y(\hat{y}(n'),n'))f^n(n') - \gamma_{ny}(\hat{y}(n'),n')\varphi(n')\right] dn' \leq 0 \qquad (5.11)$$

for all $n$. If the inequality in (5.11) is strict, then $\hat{y}(\cdot)$ is constant, i.e., there is bunching, on a neigbourhood of $n$. If $\hat{y}(\cdot)$ is strictly increasing at $n$, the inequality in (5.11) must hold as an equation. If $\hat{y}(\cdot)$ is strictly increasing on a neighbourhood of $n$, the derivative of the integral at $n$ is equal to zero, and one must have

$$\lambda(1-\gamma_y(\hat{y}(n),n))\ f^n(n) = \gamma_{ny}(\hat{y}(n),n)\varphi(n). \qquad (5.12)$$

These conditions are the same as conditions (4.18) and (4.19) above, except that $\varphi(n)$ now is given by (5.9) rather than $(\lambda-1)(1-F^n(n))$.

## 5.2 Affiliatedness

The optimality conditions in Lemma 5.1 have the same formal structure as the corresponding conditions in the unidimensional utilitarian public-good provision problem and the unidimensional utilitarian income tax problem.[19] The multidimensional nature of the problem does however appear in the terms representing the marginal social welfare that is attached to an additional unit of private-good consumption for a person with given productivity and taste parameters. These terms depend in a nontrivial way on *all* the parameters $n$ and $\theta_1, ...\theta_m$. The conditions for public good $i$, i.e., (5.2) and (5.3) therefore require taking expectations with respect to those parameters that are not directly relevant for public good $i$, i.e., $n$ and $\boldsymbol{\theta}_{-i}$, i.e. with respect to $n$ and $\boldsymbol{\theta}_{-i}$; similarly, (5.9) involves taking expectations with respect to the vector $\boldsymbol{\theta}$ of parameters not directly relevant for labour-leisure choices.

If the different parameters are mutually independent, i.e. if $F$ takes the form of a product $F^n \times F^1 \times ... \times F^m$, this integration has no effect on the underlying tradeoffs, and the first-order conditions have *exactly* the same structure as in the corresponding unidimensional problems. However, if the different parameters are *not* independent, the underlying tradeoffs are affected by the correlations.

To see why, consider the standard argument for the positivity of the optimal marginal income tax rate at $\hat{y}(n)$ for $n \in (0,1)$. If $\hat{y}(\cdot)$ is strictly

[19] For the utilitarian income tax problem, see Mirrlees (1971, 1976), Seade (1977, 1982), Ebert (1992), Brunner (1993), and, most recently, Hellwig (2007 b). For the unidimensional utilitarian public-good provision problem, see Hellwig (2005).

increasing in a neighbourhood of $n$, then, by standard arguments, (5.12) implies $\lambda T'(\hat{y}(n))f^n(n) = \gamma_{ny}(\hat{y}(n), n)\varphi(n)$. Undere the single-crossing condition $\gamma_{ny} < 0$, it follows that $T'(\hat{y}(n)) > 0$ if $\varphi(n) < 0$. For the unidimensional case, where $V_v \equiv W'$, negativity of $\varphi(n)$ follows from (5.9) and the monotonicity of $W'$. In the multidimensional setting considered here, something more is needed. If one rewrites (5.9) in the form

$$\varphi(n) = \int_n^1 \left[ \int_{[0,1]^m} W' \left( \bar{v}(n) + \sum_{j=1}^{m} \max(\theta_j - \hat{\theta}_j, 0) Q_j \right) dF(\boldsymbol{\theta}|n') - \lambda \right] dF^n(n'),$$

one sees that the monotonicity properties of the function $n' \to \int_{[0,1]^m} W' dF(\boldsymbol{\theta}|n')$ depend not only on the monotonicity properties of $W'(\cdot)$ and $\bar{v}(\cdot)$, but also on the behaviour of the conditional distribution $F(\cdot|n')$. The following, admittedly degenerate, example shows that this can make a difference for optimal tax policy.

**Example 5.2** *Assume that, for some constant* $V$, *the distribution* $F$ *is concentrated on the set*

$$\{(n, \boldsymbol{\theta})| \ \max_y [y - \gamma(y, n)] + \sum_{i=1}^{m} \theta_i Q_i^* = V\}, \tag{5.13}$$

*where* $Q_1^*, ..., Q_m^*$ *are the first-best public-good provision levels. Then any optimal utilitarian allocation satisfies* $Q_i = Q_i^*$ *and* $\hat{\theta}_i = 0$ *for all* $i$ *and* $\hat{y}(n) = \arg\max_y [y - \gamma(y, n)]$ *for all* $n$. *Public-goods finance is provided by a lump-sum tax.*

In this example, $n$ is negatively related to the taste parameters $\theta_1, .., \theta_m$ so that the differences in these parameters across people cancel out, all achieve the same utility level, and there is no reason for redistributive intervention. The example violates the assumption that the distribution $F$ has a density and therefore is not quite legitimate. However, this is merely a matter of expositional simplicity. It is not difficult to construct a distribution with a density so that $n$ is again negatively correlated with $\theta_1, .., \theta_m$ in such a way that the term $V_v$ in the integrand in (5.9) is constant and everywhere equal to $\lambda$, and, therefore, the optimal marginal income tax is everywhere equal to zero.

To exclude this possibility, I will assume that the productivity and taste parameters $n$ and $\theta_1, .., \theta_m$ are *affiliated* in the sense of Milgrom and Weber

(1982). Thus, I assume that the joint density $f$ of $n$ and $\theta_1, .., \theta_m$ satisfies the inequality

$$f((n, \boldsymbol{\theta}) \vee (n', \boldsymbol{\theta}')) \cdot f((n, \boldsymbol{\theta}) \wedge (n', \boldsymbol{\theta}')) \geq f((n, \boldsymbol{\theta})) \cdot f((n', \boldsymbol{\theta}'))$$

for all $(n, \boldsymbol{\theta})$ and $(n', \boldsymbol{\theta}')$ in $[0, 1]^{m+1}$, where $(n, \boldsymbol{\theta}) \vee (n', \boldsymbol{\theta}')$ and $(n, \boldsymbol{\theta}) \wedge (n', \boldsymbol{\theta}')$ refer to the vectors of component-wise maxima and component-wise minima of $(n, \boldsymbol{\theta})$ and $(n', \boldsymbol{\theta}')$. This assumption ensures that the parameters $n$ and $\theta_1, .., \theta_m$ are independent or positively correlated. Its implications for social marginal valuations are given in:

**Proposition 5.3** *Let $W$ be strictly concave and assume that the productivity and taste parameters $n$ and $\theta_1, .., \theta_m$ are affiliated. Then, if $\bar{v}(\cdot)$ is a strictly increasing function, and, for $i = 1, ..., m$, let $Q_i > 0$ and $\hat{\theta}_i < 1$, the maps*

$$n \to \int_{[0,1]^m} W' \left( \bar{v}(n) + \sum_{j=1}^{m} \max(\theta_j - \hat{\theta}_j, 0) Q_j \right) dF(\boldsymbol{\theta} | \tilde{n} = n) \qquad (5.14)$$

*and*

$$\theta_i \to \int_{[0,1]^m} W' \left( \bar{v}(n) + \sum_{j=1}^{m} \max(\theta_j - \hat{\theta}_j, 0) Q_j \right) dF(n, \boldsymbol{\theta}_{-i} | \tilde{\theta}_i = \theta_i), \quad (5.15)$$

*$i = 1, ..., m$, are stictly decreasing.*

## 5.3 Optimal Utilitarian Allocations under Affiliatedness

### 5.3.1 Income Taxation

If productivity and taste parameters are affiliated, optimal utilitarian income tax schedules exhibit the same features as in traditional unidimensional models. The optimal marginal income tax is positive in the interior of the income range and zero at the top. By contrast to the optimal income tax in Section 4, in the present context, it can also be desirable to have the marginal income tax rate be zero at the bottom. The difference reflects the difference in roles of the Lagrange multipler $\lambda$ in the different settings.

**Proposition 5.4** *If the productivity and taste parameters $n$ and $\theta_1, .., \theta_m$ are affiliated, the output provision function $\hat{y}(\cdot)$ that is associated with an optimal utiliarian allocation satisfies*

$$\hat{y}(n) < \arg\max_{y} [y - \gamma(y, n)] \qquad (5.16)$$

*for all* $n \in (0,1)$ *and*

$$\lim_{n\uparrow 1} \hat{y}(n) = \arg\max_y [y - \gamma(y,1)]. \tag{5.17}$$

*Moreover,*

$$\lim_{n\downarrow 0} \hat{y}(n) = \arg\max_y [y - \gamma(y,0)] \tag{5.18}$$

*unless the monotonicity constraint on* $\hat{y}(\cdot)$ *is binding in a neighbourhood of* $n = 0$.

**Corollary 5.5** *The tax schedule* $T(\cdot)$ *that is associated with the optimal utilitarian allocation in Proposition 5.4 is strictly increasing on the range of* $\hat{y}(\cdot)$, *i.e.,* $\hat{y}(n) > \hat{y}(\bar{n})$ *implies* $T(\hat{y}(n)) > T(\hat{y}(\bar{n}))$. *If* $n$ *is such that, on a neighbourhood of* $n$, $\hat{y}(\cdot)$ *is continuous and strictly increasing, then, on this neighbourhood, the schedule* $T(\cdot)$ *is continuously differentiable, with*

$$T'(\hat{y}(n')) = 1 - \gamma_y(\hat{y}(n'), n') \in (0,1) \tag{5.19}$$

*and,* $\lim_{y\uparrow\hat{y}(1)} T'(y) = 0$; *moreover,* $\lim_{y\downarrow\hat{y}(0)} T'(y) = 0$ *unless the monotonicity constraint on* $\hat{y}(\cdot)$ *is binding in a neighbourhood of* $n = 0$.

### 5.3.2 Public-Goods Provision

With affiliatedness, an optimal utilitarian allocation also involves underprovision of public goods relative to a first-best allocation.

**Proposition 5.6** *If the productivity and taste parameters* $n$ *and* $\theta_1, .., \theta_m$ *are affiliated, an optimal utilitarian allocation satisfies*

$$0 < Q_i < Q_i^* \tag{5.20}$$

*for all* $i$.

As in part (b) of Proposition 4.2, $Q_i$ is less than $Q_i^*$ because the social marginal benefits of an increase in $Q_i$ are less than $\lambda$ times $\int_0^1 \theta_i dF^i(\theta_i)$, the social marginal benefits in a first-best setting. The economic rationale is somewhat different though in the two cases. In part (b) of Proposition 4.2, the discrepancy between the social marginal benefits of an increase in $Q_i$ and the product of $\lambda$ and $\int_0^1 \theta_i dF^i(\theta_i)$ is primarily due to the fact that $\lambda$ is greater than one. Because the participation constraint is binding and public-goods provision is financed by distortionary instruments, the marginal costs of an increase in $Q_i$ are given more weight than the marginal benefits. By contrast, in Proposition 5.6, the discrepancy between the social marginal benefits of an increase in $Q_i$ and the product of $\lambda$ and $\int_0^1 \theta_i dF^i(\theta_i)$ is primarily[20] due to the fact that, with inequality aversion and affiliatedness, there is a negative correlation between the taste parameter $\tilde{\theta}_i$ and the conditional expectation, given $\tilde{\theta}_i$, of the slope $W'$ of the welfare function by which the marginal benefit of an increase in $Q_i$ to a person with taste parameter $\tilde{\theta}_i$ is weighted. This negative correlation implies that, as one aggregates across people with different tastes for the public good, high realizations of $\tilde{\theta}_i$ receive relatively lower weights than low realizations of $\tilde{\theta}_i$.[21]

### 5.3.3 Admission Fees

If one rewrites the optimality condition (5.3) in the form

$$\begin{aligned}
\lambda p_i f_i(\frac{p_i}{Q_i}))\frac{1}{Q_i} &= \lambda(1 - F_i(\frac{p_i}{Q_i})) \\
&\quad - \int_{\frac{p_i}{Q_i}} \int_{[0,1]^m} W'\left(\bar{v}(n) + \sum_{j=1}^{m} \max(\theta_j Q_j - p_j, 0)\right) dF(n, \boldsymbol{\theta}_{-i}, \theta_i),
\end{aligned} \tag{5.21}$$

one sees that, here as in Section 4, the choice of admission fees for excludable public goods involves a tradeoff between an allocative effect and a distributive effect of a price increase. The allocative effect, which is represented by the left-hand side of (5.21), is the same as before, in condition (4.28). An increase in $p_i$ induces people who are on the margin of indifference to cease asking for admission to the public good. The distributive effect arises because a price increase makes people with $\theta_i > \frac{p_i}{Q_i}$ pay more, and the additional proceeds can be used for redistribution; this effect is represented by the right-hand side of (5.21). Whereas in (4.28) the distributive effect

[20] If the admission fee $p_i$ is positive, there is an additional effect. In this case, in *both,* Propositions 4.2 *and* Proposition 5.6, some of the discrepancy between the social marginal benefits of an increase in $Q_i$ and the product of $\lambda$ and $\int_0^1 \theta_i dF^i(\theta_i)$ is due to the fact that people who do not pay the fee do not benefit from the increase in $Q_i$.

[21] If $\tilde{n}, \tilde{\theta}_1, ..., \tilde{\theta}_m$ are positively correlated, the negative correlation between $\tilde{\theta}_i$ and $W'$ also reflects the mechanism designer's aversion against inequality of payoffs induced by differences in earning abilities. As similar effect is discussed by Boadway and Keen (1993) for a model involving nonseparable utility functions. In the present model, with additively separable payoffs, correlations between the parameters $\tilde{n}, \tilde{\theta}_1, ..., \tilde{\theta}_m$ take the place of nonseparabilities in Boadway and Keen (1993). Independence of $\tilde{n}, \tilde{\theta}_1, ..., \tilde{\theta}_m$ corresponds to the separable specifications in Boadway and Keen (1993), as well as Christiansen (1981). However, because of the nondegeneracy of $\tilde{\theta}_i^h$ and inequality aversion give rise to an additional distributive concern, independence does not yield first-best provision levels.

was assessed merely in terms of the difference $\lambda - 1$ and in terms of the number of people affected, now one has to take account of the fact that the payment increases of people with different $\theta_i$ above $\frac{p_i}{Q_i}$ receive different welfare weights $W'\left(\bar{v}(n) + \sum_j \max(\theta_j Q_j - p_j, 0)\right)$; moreover, these welfare weights depend on $n$ and $\boldsymbol{\theta}_{-i}$ as well as $\theta_i$.

Equation (5.21) shows that the distributive effect involves a welfare gain rather than a welfare loss if the additional welfare attached to a marginal increase in private good consumption is lower, on average, for people with $\theta_i > \frac{p_i}{Q_i}$ than for the population as a whole. The affiliatedness assumption guarantees that this is the case. Under this assumption, the right-hand side of (5.21) is positive unless $\hat{\theta}_i$ is zero or one, in which case it is zero.

Condition (5.21) does not necessarily call for a positive admission fee. As discussed in Hellwig (2005), this condition is always satisfied at $p_i = \hat{\theta}_i = 0$ when both the allocative and the distributive effect of a small price increase are zero, the allocative effect because the people who are turned away do not care for the public good anyway, the distributive effect because the set of people with $\theta_i > \hat{\theta}_i = 0$ coincides with the population as a whole. The desirability of positive admission fees cannot be assessed from first-order conditions alone. If one looks at second-order conditions or, more generally, the global properties of the welfare functional (5.1), one finds that *positive admission fees are undesirable* if inequality aversion, i.e. the curvature $\rho_W(v) = -\frac{W''(v)}{W'(v)}$ of the welfare function $W(\cdot)$, is uniformly small; in contrast, *positive admission fees are desirable if* inequality aversion is uniformly large.

**Proposition 5.7** *If $p_1, ..., p_m$ and $\hat{\theta}_1, ..., \hat{\theta}_m$ are the admission fees and thresholds associated with an optimal utilitarian allocation, then:*

*(a) There exists a constant $r > 0$ such that, if $\rho_W(v) \in (0, r]$ for all $v$, then $p_i = \hat{\theta}_i = 0$ for all $i$.*

*(b) There exists a constant $R > 0$ such that, if $\rho_W(v) \geq R$, then $p_i > 0$ and $\hat{\theta}_i > 0$ for all $i \in J^e$. More precisely, if $\{W_k\}$ is any sequence of welfare functions such that $\lim_{k\to\infty} \rho_{W_k}(v) = \infty$, uniformly in $v$, then, for any associated sequence of optimal utilitarian allocations, the sequence $\{Q_i^k, p_i^k\}$ of provision levels and admission fees for public good $i$ converges to a limit $(Q_i^\infty, p_i^\infty)$ such that, for $i \in J^e$, the pair $(Q_i^\infty, p_i^\infty)$ is a solution to the monopoly problem $\max_{Q_i, p_i}[p_i Q_i(1 - F_i(\frac{p_i}{Q_i})) - K_i(Q_i)]$, and, for $i \in J^{ne}$, $Q_i^\infty = p_i^\infty = 0$.*

If inequality aversion is small, the allocative effects of admission fees outweigh the distribution effects, and, in the absence of participation constraints, it is undesirable to have positive admission fees for excludable public goods.[22] If inequality aversion is large, optimal utilitarian allocations must be close to Rawlsian allocations, which maximize the payoff $\bar{v}(0) = \hat{c}(0) - \gamma(\hat{y}(0), 0)$ of the worst-off person in the economy. In a Rawlsian allocation, public-goods provision is managed as a profit-maximizing monopoly so as to maximize the amount that is made available for redistribution to the people who are worst off.

Proposition 5.7 also shows that, in the absence of participation constraints, financing considerations play hardly any role for admission fees. If inequality aversion is small, they are zero, so all public-goods finance comes from direct taxes. If inequality aversion is large, admission fees are set so that public-goods provision earns a profit, which can be used for redistribution. The difference has to do with the tradeoff between distributive and allocative effects; the relation between the revenues from admission fees and the costs of public-goods provision is irrelevant.

### 5.4 Utilitarian Redistribution with Participation Constraints

How are the preceding results affected if the inequality-averse mechanism designer has to respect a participation constraint? Equivalently, how are the results of Section 4 affected if the mechanism designer is inequality averse?

One easily sees that the addition of the participation constraint (4.3) affects the optimality conditions for utilitarian redistribution *only* through the transversality condition for the boundary value $\bar{v}(0)$ of the state variable $\bar{v}$. In statement (c) of Lemma 5.1, the transversality condition $\varphi(0) = 0$ is replaced by the conditions

$$\varphi(0) \leq 0 \quad \text{and} \quad \varphi(0)\bar{v}(0) = 0, \tag{5.22}$$

which are familiar from statement (c) of Lemma 4.1. As a result of this change, one finds that the Lagrange multiplier of the feasibility/government budget constraint must satisfy the inequality

$$\lambda \geq \int_{[0,1]^{m+1}} W' \left( \bar{v}(n) + \sum_{j=1}^{m} \max(\theta_j - \hat{\theta}_j, 0) Q_j \right) dF(n, \boldsymbol{\theta}); \tag{5.23}$$

[22] A referee has pointed out that this result is not robust to a change in the assumption that, because of nonrivalry, the marginal cost of use is zero. If the marginal cost of use is positive, then, starting from a price that is equal to this marginal cost, the redistributive effect of small price increase will provide a first-order gain because, in this case, there is a significant difference between the population that pays the fee and the population that benefits from the redistribution. By standard arguments, this gain from the redistributive effect outweighs the efficiency loss from the price increase.

moreover, the inequality in (5.23) is an equation if the participation constraint is not binding, i.e., if $\bar{v}(0) > 0$.

The properties of an optimal allocation now depend on whether condition (5.23) holds as an equation or as a strict inequality. If condition (5.23) holds as an equation, the allocation will coincide with the allocation that is optimal when there is no participation constraint, and will exhibit the same properties, as laid out in Propositions 5.6 - 5.4.

A little reflection shows that this will always be the case if $\sum_i K_i(Q_i^*) \leq \max_y[y - \gamma(y, 0)]$, i.e., if, in the absence of inequality aversion, a first-best allocation is compatible with individual rationality. Since inequality aversion favours redistribution towards people with $n = 0$, an optimal allocation with inequality aversion is also compatible with with individual rationality.

Similarly, part (b) of Proposition 5.7 implies that condition (5.23) always holds as an equation if inequality aversion is uniformly large. In this case, admission fees are more than enough to cover the costs of public goods; moreover, profits from public-goods and income taxes are all used to redistribute resources towards people with $n = 0$. Given this redistribution, one must have $\bar{v}(0) > 0$.

By contrast, if $\sum_i K_i(Q_i^*) > \max_y[y - \gamma(y, 0)]$ and if inequality aversion is uniformly small, the participation constraint must be binding and the inequality in (5.23) must be strict. In this case, the optimal utilitarian allocation of Propositions 5.6 - 5.4 involves zero admission fees, and the redistribution through income taxation is insufficient to raise the payoff of people with $n = 0$ from $\max_y[y - \gamma(y, 0)] - \sum_i K_i(Q_i^*)$ to zero.

If the inequality in (5.23) is strict, the optimal allocation will exhibit roughly the same properties as the optimal admissible allocation in Section 4. In particular, admission fees for excludable public goods will all be positive so that the efficiency losses from distortionary public-goods finance are spread as widely as possible.

However, with inequality aversion, the optimality condition for admission fees takes the form (5.3), for the appropriate value of $\lambda$, rather than (4.11). The difference between these conditions corresponds to the difference between *weighted* and simple inverse-elasticicities rules in the Ramsey-Boiteux approach; see, e.g., Diamond-Mirrlees (1971). Differences in admission fees across the different public goods reflect not only differences in demand elasticities, but also differences in the weights that are given to the resource losses of people who would be hit by increases in these prices.

# A Appendix: Proofs

Lemma 3.1 is practically the same as Lemma 3.1 in Hellwig (2007 a). Therefore the reader is referred to the proof given there.

**Proof of Proposition 3.2.** I first prove the "only if" part of the proposition. Suppose that the allocation $(\mathbf{Q}, c(\cdot,.), y(\cdot,\cdot), \chi_1(\cdot,\cdot), ..., \chi_m(\cdot,\cdot))$ is incentive-compatible and renegotiation proof. By standard arguments, due to Mirrlees (1976) and Rochet (1987), incentive compatibility implies that the expected-payoff function $v(\cdot)$ is continuous. Moreover, for any $n$, the section $v(n,\cdot)$ of $v(\cdot)$ that is determined by $n$ is continuous and convex and has partial derivatives $v_i(\cdot)$ satisfying

$$v_i(n,\boldsymbol{\theta}) = \chi_i(n,\boldsymbol{\theta})Q_i \tag{A.1}$$

for $i = 1, ..., m$ and almost all $\boldsymbol{\theta} \in [0,1]^m$. For almost every $n \in [0,1]$, (A.1) in combination with Lemma 3.1 implies that

$$v_i(n,\boldsymbol{\theta}) = 0 \ \text{ if } \ \theta_i Q_i < p_i \tag{A.2}$$

and

$$v_i(n,\boldsymbol{\theta}) = Q_i \ \text{ if } \ \theta_i Q_i > p_i \tag{A.3}$$

for $i = 1, ..., m$ and almost all $\boldsymbol{\theta} \in [0,1]^m$. By integration, it follows that, for almost every $n \in [0,1]$, one has

$$v(n,\theta_1,...,\theta_m) \equiv \bar{v}(n) + \sum_{i=1}^{m} \max(\theta_i Q_i - p_i, 0), \tag{A.4}$$

for all $\boldsymbol{\theta} \in [0,1]^m$, where $\bar{v}(\cdot) := v(\cdot,\mathbf{0})$. Because $v(\cdot)$ is continuous, (A.4) in fact holds for *all* $n \in [0,1]$ and *all* $\boldsymbol{\theta} \in [0,1]^m$.

To prove statement (b), I note that, for every $(n,\boldsymbol{\theta}) \in [0,1]^{m+1}$, (A.4) implies that $v_i(n,\boldsymbol{\theta})$ satisfies (A.2) and (A.3). Upon combining (A.2) and (A.3) with (A.1), one obtains (3.9). By the definition of $v(\cdot)$, i.e., (2.7), and statement (a) and (c), one also has

$$\begin{aligned} c(n,\boldsymbol{\theta}) - \gamma(y(n,\boldsymbol{\theta}),n) &= v(n,\boldsymbol{\theta}) - \sum_{i=1}^{m} \chi_i(n,\boldsymbol{\theta})\theta_i Q_i \\ &= \bar{v}(n) - \sum_{i=1}^{m} \chi_i(n,\boldsymbol{\theta}) p_i \end{aligned}$$

for all $n$ and $\boldsymbol{\theta}$.

For statement (c), I refer the reader to the a<rgument sketched in the text. By standard arguments, due to Mirrlees (1976), incentive compatibility implies that, for any $\boldsymbol{\theta} \in [0,1]^m$, the function $v(\cdot, \boldsymbol{\theta})$ is absolutely continuous, with Radon-Nikodym derivative $-\gamma_n(y(\cdot, \boldsymbol{\theta}), \cdot)$. By (A.4), it follows that the function $\bar{v}(\cdot)$ is also absolutely continuous, with Radon-Nikodym derivative $-\gamma_n(y(\cdot, \boldsymbol{\theta}), \cdot)$, regardless of $\boldsymbol{\theta}$. For any $\boldsymbol{\theta}$, one therefore has $\gamma_n((y(n, \boldsymbol{\theta}), n) = \gamma_n((y(n, \mathbf{0}), n)$ for almost all $n$. Because $\gamma_{ny} < 0$, it follows that, for any $\boldsymbol{\theta}$, $y(n, \boldsymbol{\theta}) = y(n, \mathbf{0})$ for almost all $n$. The desired result follows by setting $\hat{y}(\cdot) := y(\cdot, \mathbf{0})$ and $\hat{c}(\cdot) := c(\cdot, \mathbf{0})$.

Statement (d) follows directly from the definitions of $\bar{v}(\cdot), \hat{y}(\cdot), \hat{c}(\cdot)$ and from the incentive compatibility of the allocation.

The second half of the proof establishes the "if" part of the proposition. Suppose that the allocation $(\mathbf{Q}, c(\cdot, .), y(\cdot, \cdot), \chi_1(\cdot, \cdot), ..., \chi_m(\cdot, \cdot))$ satisfies statements (a) - (d). By Lemma 3.1, renegotiation proofness follows immediately from statement (b). To establish incentive compatibility, consider the payoff

$$\hat{v}(\hat{n}, \hat{\boldsymbol{\theta}} | n, \boldsymbol{\theta}) = c(\hat{n}, \hat{\boldsymbol{\theta}}) + \sum_{i=1}^{m} \chi_i(\hat{n}, \hat{\boldsymbol{\theta}}) \theta_i Q_i - \gamma(y(\hat{n}, \hat{\boldsymbol{\theta}}), n) \tag{A.5}$$

that an agent with type $(n, \boldsymbol{\theta})$ obtains if he claims to have type $(\hat{n}, \hat{\boldsymbol{\theta}})$. By (3.10), one has

$$\hat{v}(\hat{n}, \hat{\boldsymbol{\theta}} | n, \boldsymbol{\theta}) = \bar{v}(\hat{n}) + \sum_{i=1}^{m} \chi_i(\hat{n}, \hat{\boldsymbol{\theta}})(\theta_i Q_i - p_i) + \gamma(y(\hat{n}, \hat{\boldsymbol{\theta}}), \hat{n}) - \gamma(y(\hat{n}, \hat{\boldsymbol{\theta}}), n). \tag{A.6}$$

By (3.9),

$$\sum_{i=1}^{m} \chi_i(\hat{n}, \hat{\boldsymbol{\theta}})(\theta_i Q_i - p_i) \leq \sum_{i=1}^{m} \chi_i(n, \boldsymbol{\theta})(\theta_i Q_i - p_i). \tag{A.7}$$

By the monotonicity of $y(\cdot, \hat{\boldsymbol{\theta}})$, one has $y(\hat{n}, \hat{\boldsymbol{\theta}}) \gtreqless y(n', \hat{\boldsymbol{\theta}})$ as $\hat{n} \gtrless n'$. By the single-crossing condition $\gamma_{ny} < 0$, it follows that

$$\gamma(y(\hat{n}, \hat{\boldsymbol{\theta}}), \hat{n}) - \gamma(y(\hat{n}, \hat{\boldsymbol{\theta}}), n) = \int_n^{\hat{n}} \gamma_n(y(\hat{n}, \hat{\boldsymbol{\theta}}), n') dn' \leq \int_n^{\hat{n}} \gamma_n(y(n', \hat{\boldsymbol{\theta}}), n') dn'. \tag{A.8}$$

By the argument of Mirrlees (1976), statement (d) implies

$$\bar{v}(\hat{n}) = \bar{v}(n) - \int_n^{\hat{n}} \gamma_n(\hat{y}(n'), n') dn'. \tag{A.9}$$

Because $y(n', \hat{\boldsymbol{\theta}}) = \hat{y}(n')$ for almost all $n'$, it follows that

$$\bar{v}(\hat{n}) = \bar{v}(n) - \int_n^{\hat{n}} \gamma_n(y(n', \hat{\boldsymbol{\theta}}), n')dn'. \tag{A.10}$$

Upon combining (A.6) with (A.7), (A.8), and (A.10), one obtains

$$\hat{v}(\hat{n}, \hat{\boldsymbol{\theta}}|n, \boldsymbol{\theta}) \leq \bar{v}(n) + \sum_{i=1}^{m} \chi_i(n, \boldsymbol{\theta})(\theta_i Q_i - p_i) = v(n, \boldsymbol{\theta}), \tag{A.11}$$

which establishes incentive compatibility. ■

To prove Lemma 4.1, I will need the following auxiliary result, which states that, at an optimal allocation, the provision level for at least one public good is positive. This result will be used to show that the Regularity Condition in Hellwig (2008) is satisfied.

**Lemma A.1** *An optimal allocation satisfies $Q_i > 0$ for at least one public good $i$.*

**Proof.** Proceeding indirectly, suppose that the lemma is false and that one has $Q_i = 0$ for all $i$. Then the aggregate surplus (4.1) and the feasibility constraint (4.6) take the form

$$\int_0^1 W(\bar{v}(n))f^n(n)dn \tag{A.12}$$

and

$$\int_0^1 [\hat{y}(n) - \bar{v}(n) - \gamma(\hat{y}(n), n)]f^n(n)dn \geq 0; \tag{A.13}$$

a maximum of (A.12) subject to (A.13) and incentive compatibility is achieved by setting

$$\hat{y}(n) = \arg\max_y [y - \gamma(y, n)] \tag{A.14}$$

and

$$\bar{v}(n) = \hat{y}(n) - \gamma(\hat{y}(n), n) \tag{A.15}$$

for all $n$, without any redistribution.

As an alternative, consider the allocation which is obtained if one sets $Q_1 = \Delta > 0$, $p_1 = 0$, and if a linear income tax with tax rate $t(\Delta)$ serves to finance the cost $K(\Delta, \mathbf{0})$. People then choose output levels

$$\hat{y}^{t(\Delta)}(n) = \arg\max_y [(1 - t(\Delta))y - \gamma(y, n)], \tag{A.16}$$

for an aggregate surplus equal to

$$\int_0^1 [(1-t(\Delta))\hat{y}^{t(\Delta)}(n) - \gamma(\hat{y}^{t(\Delta)}(n), n)]f^n(n)dn + \int_0^1 \theta_1 \Delta f_1(\theta_1) d\theta_1 \quad (A.17)$$

The dependence of $t(\Delta)$ on $\Delta$ is given implicitly by the equation

$$t(\Delta) \int_0^1 \hat{y}^{t(\Delta)}(n) f^n(n)dn - K(\Delta, \mathbf{0}) = 0. \quad (A.18)$$

At $\Delta = 0$, a solution to (A.18) is given by $t(\Delta) = 0$. At $t = 0$, one has

$$\frac{\partial}{\partial t}[t \int_0^1 \hat{y}^t(n) f^n(n)dn - K(\Delta, \mathbf{0})] = \int_0^1 \hat{y}^0(n) f^n(n)dn > 0. \quad (A.19)$$

By the implicit function theorem, it follows that, on any sufficiently small neighbourhood of $\Delta = 0$, equation (A.18) defines a continuously differentiable function $\Delta \to t(\Delta)$, with

$$\frac{dt}{d\Delta}(0) = \frac{K_1(0, \mathbf{0})}{\int_0^1 \hat{y}^0(n) f^n(n)dn}. \quad (A.20)$$

Taking account of (A.18), (A.16), and (A.20), one computes

$$\begin{aligned}
& \frac{d}{d\Delta}\left[\int_0^1 [(1-t(\Delta))\hat{y}^{t(\Delta)}(n) - \gamma(\hat{y}^{t(\Delta)}(n), n)]f^n(n)dn + \int_0^1 \theta_1 \Delta f_1(\theta_1) d\theta_1\right](0) \\
= \; & \frac{d}{d\Delta}\int_0^1 [\hat{y}^{t(\Delta)}(n) - \gamma(\hat{y}^{t(\Delta)}(n), n)]f^n(n)dn - K_1(0, \mathbf{0}) + \int_0^1 \theta_1 f_1(\theta_1) d\theta_1 \\
= \; & \int_0^1 \theta_1 f_1(\theta_1) d\theta_1 > 0 \qquad (A.21)
\end{aligned}$$

for the derivative of the aggregate surplus (A.17) with respect to $\Delta$ at $\Delta = 0$. Thus, if $Q_i = 0$ for all $i$, a small increase in $Q_1$ that is financed by a linear income tax will raise aggregate surplus (without violating incentive compatibility of individual rationality) contrary to the assumption that $Q_i = 0$ for all $i$ is optimal. ■

**Proof of Lemma 4.1.** The lemma will be proved as an instance of Theorem 5.1 in Hellwig (2008). The bring the optimization problem in the form that is assumed in that theorem, I introduce the notation

$$G_1(Q_1, ..., Q_m; \hat{\theta}_1, ..., \hat{\theta}_m) := \sum_{i=1}^m \int_{\hat{\theta}_i}^1 (\theta_i - \hat{\theta}_i) Q_i \; dF_i(\theta_i), \quad (A.22)$$

$$G_2(Q_1, ..., Q_m; \hat{\theta}_1, ..., \hat{\theta}_m) := \sum_{i=1}^{m} \hat{\theta}_i Q_i (1 - F^i(\hat{\theta}_i)) - K(Q_1, ..., Q_m). \quad (A.23)$$

Given this notation, one easily verifies that the problem of maximizing (4.1) over the set of feasible, incentive-compatible, rengotiation proof, and individually rational allocations is equivalent to the problem of choosing functions $Q_1(\cdot), ..., Q_m(\cdot), \hat{\theta}_1(\cdot), ..., \hat{\theta}_m(\cdot)$, $\bar{v}(\cdot)$,and $\hat{y}(\cdot)$ on $[0, 1]$ so as to maximize the objective

$$\int_0^1 [\bar{v}(n) + G_1(Q_1(n), ..., Q_m(n); \hat{\theta}_1(n), ..., \hat{\theta}_m(n))] f^n(n) dn \quad (A.24)$$

subject to the constraints that

$$\int_0^1 [\hat{y}(n) - \bar{v}(n) - \gamma(\hat{y}(n), n) + G_2(Q_1(n), ..., Q_m(n); \hat{\theta}_1(n), ..., \hat{\theta}_m(n))] f^n(n) dn \geq 0 \quad (A.25)$$

and, for almost all $n$,

$$Q_i'(n) = 0 \;\; \text{for all } i, \quad (A.26)$$

$$\hat{\theta}_i'(n) = 0 \;\; \text{for all } i, \quad (A.27)$$

$$\bar{v}'(n) = -\gamma_n(\hat{y}(n), n), \quad (A.28)$$

that

$$Q_i(0) \geq 0 \;\; \text{for all } i, \quad (A.29)$$

$$\hat{\theta}_i(0) = 0 \;\; \text{for all } i \in J^{ne}, \quad (A.30)$$

$$\bar{v}(0) \geq 0, \quad (A.31)$$

$$\hat{y}(0) \geq 0, \quad (A.32)$$

and, finally, that

$$\hat{y}(\cdot) \;\; \text{is a nondecreasing function.} \quad (A.33)$$

In a second step, the integral constraint (A.25) is replaced by the requirement that the variable $B(\cdot)$ satisfying

$$B(0) = 0, \quad (A.34)$$

and

$$B'(n) = [\hat{y}(n) - \bar{v}(n) - \gamma(\hat{y}(n), n) + G_2(Q_1(n), ..., Q_m(n); \hat{\theta}_1(n), ..., \hat{\theta}_m(n))] f^n(n) \quad (A.35)$$

for all $n$ must also satisfy the boundary condition

$$B(1) \geq 0. \tag{A.36}$$

The problem of maximizing (A.24) subject to (A.26) - (A.36) has the form assumed in Theorem 5.1 of Hellwig (2008). To apply this theorem, one must, however, verify the *Regularity Condition,* which requires that (i) there is at least one state variable which is not subject to an endpoint constraint at $n = 1$ and that (ii) a change in the initial value of this state variable in an admissible direction at $n = 0$ has a first-order positive effect on the endpoint constraints to which the other state variables are subjected at $n = 1$. In the present context, $B(\cdot)$ is the only state variable that is subjected to explicit constraints at both endpoints. Moreover, if $Q_i(0) > 0$ for some $i$, then any small reduction in $Q_i(0)$ will have a first-order positive effect on $B(1)$ : By (A.26), a reduction in $Q_i(0)$ by $\Delta > 0$ reduces $Q_i(n)$ by the same amount $\Delta$ for all $n$; by (A.24) and (A.35), this raises $B(1)$ by approximately $\Delta K_i(\mathbf{Q})$. By Lemma A.1, it follows that the problem of maximizing (A.24) subject to (A.26) - (A.36) satisfies the conditions of Theorem 5.1 in Hellwig (2008).[23] Therefore, there exist absolutely continuous real-valued functions $\zeta_1, ...\zeta_m, \xi_1, ..., \xi_m, \varphi, \psi, \lambda$, all defined on $[0, 1]$, such that the following hold:

(a) for $i = 1, ..., m$,

$$\zeta_i'(n) = -\frac{\partial G_1}{\partial Q_i} - \lambda(n)\frac{\partial G_2}{\partial Q_i} \tag{A.37}$$

for almost all $n$; moreover,

$$\zeta_i(0) \leq 0, \ \zeta_i(0)Q_i(0) = 0, \text{ and } \zeta_i(1)) = 0; \tag{A.38}$$

(b) for $i = 1, ..., m$,

$$\xi_i'(n) = -\frac{\partial G_1}{\partial \hat{\theta}_i} - \lambda(n)\frac{\partial G_2}{\partial \hat{\theta}_i} = 0 \tag{A.39}$$

for almost all $n$; moreover,

$$\xi_i(0) = \xi_i(1)) = 0; \tag{A.40}$$

(c) for almost all $n$,

$$\varphi'(n) = -(1 - \lambda(n))f^n(n); \tag{A.41}$$

[23] Hellwig (2008) formulates the Regularity Condition in global terms, which would require that the small reduction in $Q_i(0)$ should be feasible regardless of what $Q_i(0)$ is. However, the proof of Theorem 5.1 only uses a local version, namely, starting from the given solution to th econtrol problem, the small reduction in $Q_i(0)$ is feasible.

moreover,

$$\varphi(0) \leq 0, \quad \varphi(0)\bar{v}(0) = 0, \quad \text{and} \quad \varphi(1) = 0; \tag{A.42}$$

(d) for almost all $n$,

$$\psi'(n) = -\lambda(n)(1 - \gamma_y(\hat{y}(n), n))f^n(n) + \varphi(n)\gamma_{ny}(\hat{y}(n), n)); \tag{A.43}$$

moreover,

$$\psi(0) \leq 0, \quad \psi(0)\hat{y}(0) = 0, \tag{A.44}$$

and

$$\psi(1) \geq 0, \quad \text{and} \quad \psi(1)\hat{y}(1) = 0; \tag{A.45}$$

(e) for all $n$, $\psi(n) \leq 0$; moreover, $\psi(n) = 0$ if $\hat{y}(\cdot)$ is strictly increasing at $n$.

(f) for almost all $n$,

$$\lambda'(n) = 0; \tag{A.46}$$

moreover,

$$\lambda(1) \geq 0 \quad \text{and} \quad \lambda(1)B(1) = 0; \tag{A.47}$$

From (A.46), one infers that $\lambda(n)$ is the same for all $n$, equal to a constant $\lambda$, which, by (A.47), is nonnegative. Statements (c) - (e) of the lemma thus follow from (c) - (e) above. To prove statement (a), I note that, because $\lambda(n)$ and $Q_1(n), ..., Q_m(n); \hat{\theta}_1(n), ..., \hat{\theta}_m(n)$ are independent of $n$, (A.37) implies that, for any $i$, $\zeta_i'(n)$ is also independent of $n$, and, therefore, that

$$\zeta_i(1) - \zeta_i(0) = -\frac{\partial G_1}{\partial Q_i} - \lambda\frac{\partial G_2}{\partial Q_i}. \tag{A.48}$$

By (A.38), it follows that, for any $i$,

$$\frac{\partial G_1}{\partial Q_i} + \lambda\frac{\partial G_2}{\partial Q_i} \leq 0 \quad \text{and} \quad \left[\frac{\partial G_1}{\partial Q_i} + \lambda\frac{\partial G_2}{\partial Q_i}\right] Q_i = 0. \tag{A.49}$$

By (A.22) and (A.23), one also has

$$\frac{\partial G_1}{\partial Q_i} + \lambda\frac{\partial G_2}{\partial Q_i} \geq -\lambda K_i'(Q_i). \tag{A.50}$$

Since $Q_i = 0$ implies $K_i'(Q_i) = 0$, (A.49) and (A.50) together imply that

$$\frac{\partial G_1}{\partial Q_i} + \lambda\frac{\partial G_2}{\partial Q_i} = 0, \tag{A.51}$$

regardless of whether $Q_i > 0$ or $Q_i = 0$. Statement (a) of the lemma follows by computation of the derivatives. By a precisely analogous argument, using (A.39) and (A.40) to obtain

$$\frac{\partial G_1}{\partial \hat{\theta}_i} + \lambda \frac{\partial G_2}{\partial \hat{\theta}_i} = 0, \tag{A.52}$$

one also obtains

$$(\lambda - 1) Q_i (1 - F_i(\hat{\theta}_i)) - \lambda \hat{\theta}_i Q_i f_i(\hat{\theta}_i) = 0, \tag{A.53}$$

so statement (b) of the lemma follows if $Q_i > 0$. To see that this is indeed the case, observe that, if it was optimal to have $Q_i = 0$, then the choice of $\hat{\theta}_i$ would be a matter of indifference; in particular, it would also be optimal to set $\hat{\theta}_i = 0$. However, for $\hat{\theta}_i = 0$ and $Q_i = 0$, (4.10) cannot hold because the left-hand side is positive and the right-hand side is zero. ■

**Proof of Proposition 4.2 (a).** The allocation specified in statement (a) of the proposition maximizes aggregate surplus over the set of feasible allocations. Trivially, this allocation is also incentive-compatible and renegotiation proof. Moreover, if $K(\mathbf{Q}^*) \leq \max_y[y - \gamma(y, 0)]$, it is compatible with individual rationality. Any optimal admissible allocation must therefore be first-best. In particular, it must satisfy $Q_i = Q_i^*$ and $p_i = 0$ for all $i$, as well as (4.21). ■

**Lemma A.2** *If $K(\mathbf{Q}^*) > \max_y[y - \gamma(y, 0)]$, then the Lagrange multiplier $\lambda$ that is associated with an optimal admissible allocation satifies $\lambda > 1$.*

**Proof.** I claim that, if $\lambda = 1$, one must have $T(\hat{y}(0)) \geq K(\mathbf{Q}^*)$ for all $n$, hence,

$$\bar{v}(0) \leq \hat{y}(0) - K(\mathbf{Q}^*) - \gamma(\hat{y}(0), 0),$$

which is nonnegative only if $K(\mathbf{Q}^*) \leq \hat{y}(0) - \gamma(\hat{y}(0), 0)$. Thus, $K(\mathbf{Q}^*) > \max_y[y - \gamma(y, 0)]$ is incompatible with $\lambda = 1$. Because, as discussed in the text, conditions (4.12) and (**??**) also rule out the possibility that $\lambda < 1$, the lemma then follows.

To prove that $\lambda = 1$ implies $T(\hat{y}(0)) \geq K(\mathbf{Q}^*)$, I first show that, if $\lambda = 1$, then $T(\hat{y}(n))$ is same for all $n$. For any $n$ and $\bar{n}$, one has

$$\begin{aligned} & T(\hat{y}(n)) - T(\hat{y}(\bar{n})) = \hat{y}(n) - \hat{y}(\bar{n}) - (\hat{c}(n) - \hat{c}(\bar{n})) \\ = \; & \hat{y}(n) - \hat{y}(\bar{n}) - (\bar{v}(n) - \bar{v}(\bar{n})) - (\gamma(\hat{y}(n), n) - \gamma(\hat{y}(\bar{n}), \bar{n}) \\ = \; & \int_{\bar{n}}^{n} (1 - \gamma_y(\hat{y}(n'), n')) d\hat{y}(n') - \int_{\bar{n}}^{n} d\bar{v}(n') - \int_{\bar{n}}^{n} \gamma_n(\hat{y}(n'), n') d\hat{y}(n') \end{aligned} \tag{A.54}$$

By the incentive compatibility condition (4.7), the last two terms cancel, and one obtains

$$T(\hat{y}(n)) - T(\hat{y}(\bar{n})) = \int_{\bar{n}}^{n} (1 - \gamma_y(\hat{y}(n'), n'))d\hat{y}(n'). \tag{A.55}$$

To prove that $T(\hat{y}(n))$ is same for all $n$, it thus suffices to show that, if $\lambda = 1$, then $\gamma_y(\hat{y}(n'), n') = 1$ for all $n'$.

If $\lambda = 1$, the optimality condition (4.17) takes the form

$$\int_{n}^{1} (1 - \gamma_y(\hat{y}(n'), n'))f^n(n')dn' \leq 0 \tag{A.56}$$

for all $n$. I claim that this inequality must in fact hold as an equation for all $n$. For suppose that the inequality in (A.56) was strict for some $n$. Then $n$ would be part of an interval $(n_0, n_1)$ on which the monotonicity constraint on $\hat{y}(\cdot)$ is binding, and one should have $\hat{y}(n') = \hat{y}(n)$ for all $n' \in (n_0, n_1)$.

If $\hat{y}(n) > 0$, one must have

$$\int_{n_0}^{1} (1 - \gamma_y(\hat{y}(n'), n'))f^n(n')dn' = 0, \tag{A.57}$$

regardless of whether $n_0 > 0$ or $n_0 = 0$. If $n_0 > 0$, (A.57) holds because, at an endpoint of a bunching interval, $\hat{y}(\cdot)$ is strictly increasing. If $n_0 = 0$, (A.57) follows from the transversality condition (4.14). Now conditions (A.57) and (A.56) jointly imply that, at $n = n_0$, the integral on the left-hand side of (A.56) is nonincreasing in $n$. Hence, $1 - \gamma_y(\hat{y}(n), n_0) \geq 0$. Because $\gamma_{yn} < 0$, it follows that $1 - \gamma_y(\hat{y}(n), n') > 0$ for all $n' > n_0$. Therefore,

$$\int_{n_0}^{n_1} (1 - \gamma_y(\hat{y}(n'), n'))f^n(n')dn' > 0. \tag{A.58}$$

However, (A.57) and (A.58) together yield

$$\int_{n_1}^{1} (1 - \gamma_y(\hat{y}(n'), n'))f^n(n')dn' < 0. \tag{A.59}$$

(A.59) implies that $n_1 < 1$ and that $\hat{y}(\cdot)$ is constant in a neighbourhood of $n_1$. This contradicts the definition of $n_1$ as the upper endpoint of a bunching interval.

Alternatively, if $\hat{y}(n) = 0$, one has $n_0 = 0$ and $\gamma_y(\hat{y}(n), n') = 0$ for all $n'$. Again, one obtains (A.58). Upon combining this condition with (A.56) (for $n = n_0$), one again obtains (A.59). By the same argument as before, this yields is incompatible with the assumption that $n_1$ is the upper endpoint of a bunching interval.

Thus, in either case, if $\hat{y}(n) > 0$ and if $\hat{y}(n) = 0$, the assumption that, for $n$, condition (A.56) holds as a strict inequality leads to a contradiction. Given that (A.56) holds as an equation for all $n$, the derivative of the left-hand side with respect to $n$ must be equal to zero. This implies $\gamma_y(\hat{y}(n), n) = 1$ for all $n$ and, by (A.55), $T(\hat{y}(n)) - T(\hat{y}(\bar{n})) = 0$ for all $n$ and $\bar{n}$.

By (4.11) and (4.10), $\lambda = 1$ implies $\hat{\theta}_i = 0$ and $Q_i = Q_i^*$ for all $i$, as well as $T(\hat{y}(n)) = T(\hat{y}(0))$ for all $n$. By the feasibility constraint (3.22), it follows that $\lambda = 1$ implies $T(\hat{y}(0)) \geq K(\mathbf{Q}^*)$. ∎

**Proof of Proposition 4.2 (b).** By Lemma A.2, $K(\mathbf{Q}^*) > \max_y[y - \gamma(y, 0)]$ implies $\lambda > 1$. By (4.11), one it follows that $\hat{\theta}_i \in (0, 1)$, hence $p_i \in (0, Q_i)$ for $i \in J^e$. With $\lambda > 1$ and $\hat{\theta}_i \in [0, 1)$ for all $i$, one then also has

$$0 < \frac{1}{\lambda} \int_{\hat{\theta}_i}^{1} (\theta_i - \hat{\theta}_i) dF_i(\theta_i) + \hat{\theta}_i (1 - F_i(\hat{\theta}_i)) < \int_0^1 \theta_i dF_i(\theta_i), \tag{A.60}$$

so (4.10) implies (4.22).

I next show that $\hat{y}(n) < \arg\max_y[y - \gamma(y, n)]$ or, equivalently, that

$$\gamma_y(\hat{y}(n), n) < 1 \tag{A.61}$$

for all $n$. If $n$ is such that (4.17) holds with equality, one must have

$$\int_n^{\hat{n}} \left[\lambda(1 - \gamma_y(\hat{y}(n'), n')) f^n(n') + \gamma_{ny}(\hat{y}(n'), n')(\lambda - 1)(1 - F^n(n'))\right] dn' \geq 0 \tag{A.62}$$

for all $\hat{n} > n$. Then there exists $n' > n$ arbitrarily close to $n$ such that

$$\lambda(1 - \gamma_y(\hat{y}(n'), n')) f^n(n') + \gamma_{ny}(\hat{y}(n'), n')(\lambda - 1)(1 - F^n(n')) \geq 0 \tag{A.63}$$

and, therefore, since $\gamma_{ny}(\hat{y}(n'), n') < 0$,

$$1 - \gamma_y(\hat{y}(n'), n') > 0. \tag{A.64}$$

Upon taking limits as $n' \downarrow n$ and noting that, by the monotonicity of $\hat{y}(\cdot)$, $\hat{y}(n) \leq \lim_{n' \downarrow n} \hat{y}(n')$, one obtains the desired result. Alternatively, if $n$ is such that (4.17) holds with a strict inequality, then $n$ is part of an interval $(n_0, n_1)$ on which the monotonicity constraint on $\hat{y}(\cdot)$ is binding, and one has $\hat{y}(n') = \hat{y}(n)$ for all $n' \in (n_0, n_1)$. If $\hat{y}(n) = 0$, one trivially has $\gamma_y(\hat{y}(n'), n') = \gamma_y(0, n') = 0 < 1$ for all $n' \in [0, n_1)$. If $\hat{y}(n) > 0$, one has $n > n_0$ and

$n_0$ is such that (4.17) holds with equality, either because $n_0$ is a point at which $\hat{y}(\cdot)$ is increasing or because $n_0 = 0$. By the argument just given, it follows that (A.64) holds for $n' > n_0$ arbitrarily close to $n_0$. Such $n'$ can be chosen to be smaller than $n$. Then, with $\hat{y}(n') = \hat{y}(n)$, the single-crossing condition $\gamma_{yn} < 0$ implies $\gamma_y(\hat{y}(n), n) < \gamma_y(\hat{y}(n'), n')$ and, therefore, by (A.64), $\gamma_y(\hat{y}(n), n) < 1$. This establishes (4.24).

To prove (4.28), observe that (4.17) implies

$$\int_n^1 \left[\lambda(1 - \gamma_y(\hat{y}(n'), n'))f^n(n')\right] dn' \leq \int_n^1 \left|\gamma_{ny}(\hat{y}(n'), n')\right| (\lambda-1)(1-F^n(n'))dn'. \tag{A.65}$$

If one divides both sides by $1 - n$ and takes limits as $n \uparrow 1$, one obtains

$$\lim_{n'\uparrow 1}(1 - \gamma_y(\hat{y}(n'), n')) = 0, \tag{A.66}$$

which is equivalent to (4.28). ■

**Proof of Corollary 4.3.** **(a)** By Proposition 4.2(a) and equation (A.55), $T(\hat{y}(n))$ is independent of $n$. Feasibility implies that the common value of $T(\hat{y}(n))$ is at least $K(\mathbf{Q}^*)$. Optimality implies that it is no more than $K(\mathbf{Q}^*)$. **(b)** By Proposition 4.2(b) and equation (A.55), the tax schedule $T(\cdot)$ is increasing on the range of $\hat{y}(\cdot)$. If $\hat{y}(\cdot)$ is continuous and strictly increasing on a neighbourhood of $n$, then, by taking (A.55) with $\hat{n} = n - \varepsilon$, dividing by $\hat{y}(n) - \hat{y}(n - \varepsilon)$, and taking limits as $\varepsilon$ goes to zero, one obtains $T'(\hat{y}(n)) = 1 - \gamma_y(\hat{y}(n), n)$. ■

Proposition 4.4 is based on the following result.

**Proposition A.3** *For any $S \geq 0$, let $Q_1(S), ..., Q_m(S)$ and $\hat{\theta}_1(S), ..., \hat{\theta}_m(S)$ be a solution to the problem of maximizing*

$$\sum_{i=1}^m \int_{\hat{\theta}_i}^1 (\theta_i - \hat{\theta}_i)Q_i dF_i(\theta_i) \tag{A.67}$$

*under the constraints that $\hat{\theta}_i = 0$ for $i \in J^{ne}$ and*

$$\sum_{i=1}^m Q_i\hat{\theta}_i(1 - F_i(\hat{\theta}_i)) + S \geq \sum_{i=1}^m K_i(Q_i). \tag{A.68}$$

*If the functions $\theta_i \to \frac{\theta_i f_i(\theta_i)}{1-F_i(\theta_i)}$, $i \in J^e$, are nondecreasing, then, for $S$ and $\bar{S}$ satisfying $0 \leq \bar{S} < S < \sum_{i=1}^m K_i(Q_i^*)$, one has $Q_i(\bar{S}) < Q_i(S)$ for all $i$ and $\hat{\theta}_i(\bar{S}) > \hat{\theta}_i(S)$ for all $i \in J^e$.*

**Proof.** For any $S$, one must have $Q_i(S) > 0$ for all $i$. For suppose that $Q_i(S) = 0$ for some $i$. Then public good $i$ makes no contribution to the objective (A.67) or to either side of the feasiblity constraint (A.68). However, if one sets $\hat{\theta}_i = \frac{1}{2}$, a small increase in $Q_i$ above zero will increase the objective (A.67), as well as the difference between the revenue side and the cost side of the constraint (A.68). The assumption that the optimal $Q_i(S)$ is zero thus leads to a contradiction and must be false.

Given that $Q_i(S) > 0$ for all $i$, the first-order conditions for $Q_i(S)$ require that

$$\int_{\hat{\theta}_i(S)}^{1} (\theta_i - \hat{\theta}_i(S))dF_i(\theta_i) + \lambda(S)\hat{\theta}_i(S)(1 - F_i(\hat{\theta}_i(S))) - \lambda(S)K_i'(Q_i(S)) = 0 \tag{A.69}$$

for all $i$ where $\lambda(S)$ is the Lagrange multiplier of the constraint (A.68). From (A.69), one immediately infers that $\hat{\theta}_i(S) < 1$ for all $i$. Therefore, first-order conditions for $\hat{\theta}_i(S)$ require that

$$-(1 - F_i(\hat{\theta}_i(S))) + \lambda(1 - F_i(\hat{\theta}_i(S)) - \hat{\theta}_i(S)f_i(\hat{\theta}_i(S))) \leq 0 \tag{A.70}$$

for all $i \in J^e$, with equality unless $\hat{\theta}_i(S) = 0$.

I claim that, if $0 \leq S < \sum_{i=1}^{m} K_i(Q_i^*)$, then one must have $\lambda(S) > 1$ and $\hat{\theta}_i(S) > 0$ for all $i \in J^e$. If $\lambda(S) \leq 1$, then (A.70) implies $\hat{\theta}_i(S) = 0$, and (A.69) implies $\int \theta_i dF_i(\theta_i) \leq K_i'(Q_i(S))$, hence $Q_i(S) \geq Q_i^*$. From the constraint (A.68), it then follows that $S \geq \sum_{i=1}^{m} K_i(Q_i^*)$.

If $0 \leq S < \sum_{i=1}^{m} K_i(Q_i^*)$, the first-order conditions (A.69) and (A.70) can therefore be rewritten as

$$\frac{1}{\lambda(S)} \int_{\hat{\theta}_i(S)}^{1} \theta_i dF_i(\theta_i) + \left(1 - \frac{1}{\lambda(S)}\right) \hat{\theta}_i(S)(1 - F_i(\hat{\theta}_i(S))) = K_i'(Q_i(S)) \tag{A.71}$$

and

$$\eta_i(\hat{\theta}_i(S)) = \frac{\lambda(S) - 1}{\lambda(S)} \tag{A.72}$$

where, for any $\theta_i$,

$$\eta_i(\theta_i) := \frac{\theta_i f_i(\theta_i)}{1 - F_i(\theta_i)}. \tag{A.73}$$

Now consider $S$ and $\bar{S}$ as specified in the proposition. Trivially, the maximized value of the objective function (A.67) must be larger for $S$ than for $\bar{S}$. By inspection of (A.67), it follows that, for at least one public good $j$, one must have $Q_j(\bar{S}) < Q_j(S)$ or $\hat{\theta}_j(\bar{S}) > \hat{\theta}_j(S)$.

Suppose first that one has $Q_j(\bar{S}) < Q_j(S)$ or $\hat{\theta}_j(\bar{S}) > \hat{\theta}_j(S)$ for some $j \in J^e$. Invoking the monotonicity assumption on $\eta_i(\cdot)$, I claim that, in fact, one must have $Q_j(\bar{S}) < Q_j(S)$ *and* $\hat{\theta}_i(\bar{S}) > \hat{\theta}_i(S)$. To prove this, I use (A.72) to rewrite (A.71) in the form

$$(1-\eta_j(\hat{\theta}_j(S))) \int_{\hat{\theta}_j(S)}^1 \theta_j dF_j(\theta_j)+\eta_j(\hat{\theta}_j(S))\hat{\theta}_j(S)(1-F_j(\hat{\theta}_j(S))) = K_j'(Q_j(S)). \tag{A.74}$$

The left-hand side of this condition depends on $S$ only through $\hat{\theta}_j(S)$. Trivially, (A.74) implies that $Q_j(\bar{S}) < Q_j(S)$ if and only if $L_j(\hat{\theta}_j(\bar{S})) < L_j(\hat{\theta}_j(S))$, where, for any $\hat{\theta}_j$,

$$L_j(\hat{\theta}_j) := (1-\eta_j(\hat{\theta}_j)) \int_{\hat{\theta}_j}^1 \theta_j dF_j(\theta_j) + \eta_j(\hat{\theta}_j)\hat{\theta}_j(1-F_j(\hat{\theta}_j)). \tag{A.75}$$

From (A.75), one obtains

$$\begin{aligned}
& L_j(\hat{\theta}_j(S)) - L_j(\hat{\theta}_j(\bar{S})) \\
= \;& \int_{\hat{\theta}_j(\bar{S})}^{\hat{\theta}_j(S)} [-(1-\eta_j(\hat{\theta}_j))\hat{\theta}_j f_i(\hat{\theta}_j) + \eta_j(\hat{\theta}_j)(1-F_j(\hat{\theta}_j) - \hat{\theta}_j f_i(\hat{\theta}_j))] d\hat{\theta}_j \\
& - \int_{\hat{\theta}_i(\bar{S})}^{\hat{\theta}_i(S)} \int_{\hat{\theta}_j}^1 (\theta_j - \hat{\theta}_j) dF_j(\theta_j) d\eta_j(\hat{\theta}_j) \\
= \;& - \int_{\hat{\theta}_i(\bar{S})}^{\hat{\theta}_i(S)} \int_{\hat{\theta}_j}^1 (\theta_j - \hat{\theta}_j) dF_j(\theta_j) d\eta_j(\hat{\theta}_j),
\end{aligned}$$

so that, if $\eta_j(\cdot)$ is an increasing function, one has $L_j(\hat{\theta}_j(\bar{S})) < L_j(\hat{\theta}_j(S))$ if and only if $\hat{\theta}_j(\bar{S}) > \hat{\theta}_j(S)$. By (A.74), it follows that $Q_j(\bar{S}) < Q_j(S)$ if and only if $\hat{\theta}_j(\bar{S}) > \hat{\theta}_j(S)$. Hence $Q_j(\bar{S}) < Q_j(S)$ *or* $\hat{\theta}_j(\bar{S}) > \hat{\theta}_j(S)$ implies $Q_j(\bar{S}) < Q_j(S)$ *and* $\hat{\theta}_j(\bar{S}) > \hat{\theta}_j(S)$.

By (A.72) and the monotonicity of $\eta_j(\cdot)$, one then also has $\eta_j(\hat{\theta}_j(S)) < \eta_j(\hat{\theta}_j(\bar{S}))$ and $\lambda(S) < \lambda(\bar{S})$. By (A.72), one then has $\eta_i(\hat{\theta}_i(S)) < \eta_i(\hat{\theta}_i(\bar{S}))$ for all $i \in J^e$. By the monotonicity of $\eta_i(\cdot)$, this implies $\hat{\theta}_i(S) < \hat{\theta}_i(\bar{S})$ for all $i \in J^e$. By the argument just given, one then also has $Q_i(S) > Q_i(\bar{S})$ for all $i \in J^e$. For $i \in J^{ne}$, $Q_i(S) > Q_i(\bar{S})$ follows from (A.71) in combination with the fact that $\hat{\theta}_i(S) = \hat{\theta}_i(\bar{S}) = 0$ and $\lambda(S) < \lambda(\bar{S})$.

Suppose, alternatively, that $Q_j(\bar{S}) < Q_j(S)$ or $\hat{\theta}_j(\bar{S}) > \hat{\theta}_j(S)$ for some $j \in J^{ne}$. Because $j \in J^{ne}$ implies $\hat{\theta}_j(\bar{S}) = \hat{\theta}_j(S) = 0$, one must actually have $Q_j(\bar{S}) < Q_j(S)$. From (A.71), one then infers that $\lambda(S) < \lambda(\bar{S})$. For $i \neq j$, the desired result follows by the same argument as before. ■

Proposition 4.4 is the special case of Proposition A.3 that is obtained by setting $\bar{S} = 0$ and $S = \int_0^1 T(\hat{y}(n))\, dF^n(n)$.

The proof of Lemma 5.1 is very similar to the proof of Lemma 4.1. The main difference is that now the state variable $\bar{v}(\cdot)$ and the potential for varying $\bar{v}(0)$ are used to verify the Regularity Condition in Hellwig (2008). The proof is given in the Supplementary Material.

**Example 5.2.** Let $Q_1, ..., Q_m, \hat{\theta}_1, ..., \hat{\theta}_m, \hat{y}(\cdot), \bar{v}(\cdot)$ correspond to an optimal utilitarian allocation. Then one has

$$\begin{aligned} & \int W\left(\bar{v}(n) + \sum_{i=1}^{m} \max(\theta_i - \hat{\theta}_i, 0) Q_i\right) dF \\ \geq \; & \int W\left(\max_y [y - \gamma(y, n)] + \sum_{i=1}^{m} \theta_i Q_i^* - \sum_{i=1}^{m} K_i(Q_i^*)\right) dF, \end{aligned}$$

the right-hand side corresponding to the first-best allocation where public goods are financed by lump-sum taxes. Because $W$ is strictly concave, it follows that

$$\int W'\left(\max_y [y - \gamma(y, n)] + \sum_{i=1}^{m} \theta_i Q_i^* - \sum_{i=1}^{m} K_i(Q_i^*)\right)$$
$$\left(\bar{v}(n) + \sum_{i=1}^{m} \max(\theta_i - \hat{\theta}_i, 0) Q_i - \max_y [y - \gamma(y, n)] - \sum_{i=1}^{m} \theta_i Q_i^* + \sum_{i=1}^{m} K_i(Q_i^*)\right) dF \geq 0.$$

Under the assumption of the example, the first term in the integrand is equal to $W'\left(\bar{v} - \sum_{i=1}^{m} K_i(Q_i^*)\right)$, $F$-almost everywhere. Therefore, one must have

$$\int \left(\bar{v}(n) + \sum_{i=1}^{m} \max(\theta_i - \hat{\theta}_i, 0) Q_i\right) dF \geq \int \left(\max_y [y - \gamma(y, n)] + \sum_{i=1}^{m} \theta_i Q_i^* - \sum_{i=1}^{m} K_i(Q_i^*)\right) dF,$$

or

$$\int_0^1 \bar{v}(n) dF^n + \sum_{i=1}^{m} \int_{\hat{\theta}_i}^1 (\theta_i - \hat{\theta}_i) Q_i dF_i \geq \int_0^1 \max_y [y - \gamma(y, n)] dF^n + \sum_{i=1}^{m} \int_0^1 \theta_i Q_i^* dF_i - \sum_{i=1}^{m} K_i(Q_i^*).$$

Because the optimal utilitarian allocation is feasible, is follows that

$$\begin{aligned} & \int_0^1 (\hat{y}(n) - \gamma(\hat{y}(n), n)) dF^n + \sum_{i=1}^{m} \int_{\hat{\theta}_i}^1 \theta_i Q_i dF_i - \sum_{i=1}^{m} K_i(Q_i) \\ \geq \; & \int_0^1 \max_y [y - \gamma(y, n)] dF^n + \sum_{i=1}^{m} \int_0^1 \theta_i Q_i^* dF_i - \sum_{i=1}^{m} K_i(Q_i^*), \end{aligned}$$

which is only possible if $Q_i = Q_i^*$ and $\hat{\theta}_i = 0$ for all $i$, and if $\hat{y}(n) - \gamma(\hat{y}(n), n) = \max_y [y - \gamma(y, n)]$ for $F^n$-almost all $n$. ■

**Proof of Proposition 5.3.** Let $n_1$ and $n_2$ be such that $n_1 < n_2$, and note that, by the concavity of $W$, the function $\boldsymbol{\theta} \to W' \left( \bar{v}(n) + \sum_{i=1}^m \max(\theta_i - \hat{\theta}_i, 0) Q_i \right)$ is nonincreasing. By Theorem 5, p. 1100, of Milgrom and Weber (1982) therefore, the affiliatedness assumption on $F$ implies that

$$\begin{aligned} & \int_{[0,1]^m} W' \left( \bar{v}(n_2) + \sum_{i=1}^m \max(\theta_i - \hat{\theta}_i, 0) Q_i \right) dF(\boldsymbol{\theta} | n_1) \\ \geq & \int_{[0,1]^m} W' \left( \bar{v}(n_2) + \sum_{i=1}^m \max(\theta_i - \hat{\theta}_i, 0) Q_i \right) dF(\boldsymbol{\theta} | n_2). \end{aligned} \quad \text{(A.76)}$$

By the strict concavity of $W$ and the strict monotonicity of $\bar{v}(\cdot)$, it follows that

$$\begin{aligned} & \int_{[0,1]^m} W' \left( \bar{v}(n_1) + \sum_{i=1}^m \max(\theta_i - \hat{\theta}_i, 0) Q_i \right) dF(\boldsymbol{\theta} | n_1) \\ > & \int_{[0,1]^m} W' \left( \bar{v}(n_2) + \sum_{i=1}^m \max(\theta_i - \hat{\theta}_i, 0) Q_i \right) dF(\boldsymbol{\theta} | n_2), \end{aligned} \quad \text{(A.77)}$$

which proves that the map (5.14) is strictly decreasing. A precisely analogous argument also establishes that the (5.15) is strictly decreasing. The details are left to the reader. ■

**Proof of Proposition 5.4.** Given the strict monotonicity of the map (5.14), there exists a unique $\hat{n} \in [0, 1]$ such that

$$V_v(\bar{v}(n'), n', Q_1, ...Q_m, \hat{\theta}_1, ..., \hat{\theta}_m) \lesseqqgtr \lambda \quad \text{as} \quad n' \gtreqqless \hat{n}. \quad \text{(A.78)}$$

From (A.78), one infers that the function $n \to \varphi(n) = \int_n^1 [V_v - \lambda] \, dF^n(n')$ is decreasing for $n < \hat{n}$ and increasing for $n > \hat{n}$. Because $\varphi(0) = \varphi(1) = 0$, it follows that $\hat{n} \in (0, 1)$ and that $\varphi(n) < 0$ for all $n \in (0, 1)$.

Now the argument for (5.16) and (5.17) is the same as in the proof of (4.24) and (4.25) in Proposition 4.2 and is left to the reader. As for (5.18), I note that, if there exists a sequence $\{n^k\}$ of points at which $\hat{y}(\cdot)$ is increasing, then by Lemma 5.1, one has

$$\int_{n^k}^1 \left[ -\lambda(1 - \gamma_y(\hat{y}(n'), n')) f^n(n') + \gamma_{ny}(\hat{y}(n'), n') \varphi(n') \right] dn' = 0 \quad \text{(A.79)}$$

for all $k$.

$$\int_{n^k}^{n^{k-1}} \left[-\lambda(1-\gamma_y(\hat{y}(n'),n'))f^n(n') + \gamma_{ny}(\hat{y}(n'),n')\varphi(n')\right] dn' = 0 \quad \text{(A.80)}$$

for all $k$. Then there exist sequences $\{\hat{n}^k\}, \{\bar{n}^k\}$ converging to zero such that for all $k$ one has

$$-\lambda(1-\gamma_y(\hat{y}(\hat{n}^k),\hat{n}^k))f^n(\hat{n}^k) + \gamma_{ny}(\hat{y}(\hat{n}^k),\hat{n}^k)\varphi(\hat{n}^k) \leq 0$$

and

$$-\lambda(1-\gamma_y(\hat{y}(\bar{n}^k),\bar{n}^k))f^n(\bar{n}^k) + \gamma_{ny}(\hat{y}(\bar{n}^k),\bar{n}^k)\varphi(\bar{n}^k) \geq 0.$$

Upon defining $\bar{y}(0) := \lim_{n\downarrow 0}\hat{y}(n)$ and taking limits as $\hat{n}^k$ and $\bar{n}^k$ go to zero, using the fact that $\varphi(\cdot)$ is continuous and $\varphi(0) = 0$, one obtains $-\lambda(1-\gamma_y(\bar{y}(0),0))f^n(0) \leq 0$ and $-\lambda(1-\gamma_y(\bar{y}(0),0))f^n(0) \geq 0$. Hence, $\gamma_y(\bar{y}(0),0) = 1$ and $\bar{y}(0) = \arg\max_y[y - \gamma(y,n)]$. ∎

Corollary 5.5 follows by the same argument as Corollary 4.3.

The proofs of Propositions 5.6 and 5.7 are similar to the proofs of the corresponding results in Hellwig (2005). They are given in the Supplementary Material.

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