The measurement of output growth and the decline of equipment prices[1]

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Summary: The permanent decline of the price of equipment relative to consumption goods renders traditional fixed-base quantity indexes obsolete, because of the well-known substitution-bias problem. National Income and Product Accounts (NIPA) responded switching to a flexible-base quantity index to measure real GDP growth. In this paper we argue that this is a correct measure of real GDP growth from the point of view of the economic theory of index numbers. In a two-sector model of endogenous growth, we use the Bellman equation to explicitly represent preferences on consumption and investment and we prove that the Fisher-Shell true quantity index is equal to the flexible-base quantity index used by NIPA. In this stylized framework, we are able to clarify some points still under debate. In particular, we show that the growth rate of consumption is not all that matters for welfare representation.

Keywords: Measurement of growth, Quantity indexes, Embodied technical progress.

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1 Introduction

The Bureau of Economic Analysis (BEA) featured in its National Income and Product Accounts (NIPA's) a fixed-base quantity index to measure real GDP growth. The fast decline of the relative prices of equipment, notably computers and peripheral equipment, has lead BEA to consider alternative measures of real growth.[2] The traditional fixed-base Laspeyres quantity index yields a reasonable measurement of real growth provided that relative prices remain stable. If the price of equipment declines, the weight of investment with respect to consumption in the Laspeyres index becomes obsolete quickly enough to have a relevant impact on growth measurement. Changing relative prices called for flexible-base quantity indexes. Since the early 1990's the NIPA's feature, together with the fixed-base index, a chained-type index built on the Fisher Ideal index.[3] National Accounts in other countries already calculated alternative measures of real growth, like a chained-type index based on the Laspeyres index in the Netherlands and Norway. European Union Member States will soon follow BEA: Commission Decision 98/715/EC establishes 2005 as the beginning of a period in which Member States will progressively adapt their National Accounts. Among these changes stands out the publication of a chained-type index based on the Fisher Ideal index. However, the theoretical legitimation of these new measures is still under debate in the growth literature.

On the other side, growth theory has been reformulated in the last decade in order to replicate the observed trend in relative prices. Based on Solow (1960), Greenwood et al (1997) propose a simple two-sector optimal growth model where productivity grows faster in the investment sector than in the consumption sector. Many other papers have follow.[4] In this family of models, as in the data, investment grows at a larger rate than consumption, which raises the fundamental problem of measuring output growth.

As showed by Felbermayr and Licandro (2004), the two-sector AK model proposed by Rebelo (1991) reproduces the observations referred to as above. In this framework, we use the economic theory of index numbers to formulate a true quantity index of output growth. The contribution of this paper is to show that the chained-type Fisher Ideal

[2] Cummins and Violante (2002) contains a thorough review of the evolution of constant-quality prices for equipment from 1947 to 2000 in the US. Since the mid-80's BEA provides with a constant-quality price index for computers and peripherals but historical series first appeared in the seminal contribution of Robert J. Gordon (1990).

[3] Young (1992) is a non-technical presentation of the methodological changes introduced in the NIPA's. Whelan (2002, 2003) is a more detailed guide into the new methods to measure real growth in use at BEA. For economic index number theory see IMF (2004, chapter 17) or Fisher and Shell (1998).

[4] List some of them.

index used by NIPA is equal to the welfare-relevant quantity index in the two-sector growth model. Exploiting the explicit utility representation of instantaneous preferences over consumption and investment, we construct the Fisher-Shell true quantity index. The true quantity index is then proved to be equal to the Divisia index which in turn is equal to the Fisher Ideal index. This means that National Accounts measure correctly real growth and, in particular, that the growth rate of investment does contain information relevant to the welfare of the representative individual. In a sense, we challenge the view that separates welfare from productivity issues. Output quantity indexes are viewed as relevant to productivity measurements while consumer price indexes are viewed as relevant for welfare (see e.g. Whelan (2002, p.222)).

In this framework, we are able to clarify the debate in the growth and growth accounting literatures. Some authors like Greenwood et al (1997) and Oulton (2002) develop a modern version of the paradigm that only consumption matters to utility. Hence, by different routes, they end up suggesting that the growth rate of consumption is all that matters. We shall discuss that their suggestion relies, at least in part, on the assumption that the consumption good is used as input in the machines sector. The Fisher-Shell true quantity index is a better measurement of real growth from a conceptual point of view and we illustrate our point with a simple example.

2 A two-sector AK model

In this section, we describe a simple version of the two-sector AK model proposed by Rebelo (1991). As pointed out by Felbermayr and Licandro (2004), it is the simplest endogenous growth that replicates the observed permanent decline in the price of equipment and the permanent increase in the equipment to output ratio. In this context, it is particularly clear that the aggregation issue is far from trivial since consumption and equipment grow at different rates.

2.1 A model of embodied technical progress

The extent to what technical progress affects all production factors (disembodied) or it is incorporated in new machines and therefore embodied in quality-adjusted productive investment (embodied) has been considered a relevant question since the seminal work of Solow (1960). Following this early contribution, embodied technical change (ETC) has been usually represented in model economies with a consumption good sector using machines as input and a machines sectors using the consumption good as input. Investment-specific technical progress is interpreted to be embodied in machines, but consumption-specific technical progress is considered to be disembodied. Two recent important contributions follow this tradition. Hulten (1992) in growth accounting and

Greenwood et al (1997) in a general equilibrium framework argue that the embodiment hypothesis is a reasonable explanation for the observed decline of equipment prices.

The model in this section is based in Rebelo (1991) and follows Licandro and Felbermayr (2004) closely. The stock of machines at each instant t is k_t, from which $b_t \leq k_t$ is devoted to the production of the consumption good

$$c_t = b_t^{\alpha},$$

where $\alpha \in (0,1)$. The remaining stock $k_t - b_t$ is devoted to the production of new machines with a linear technology

$$\dot{k}_t = A(k_t - b_t) - \delta k_t,$$

where $A > 0$, while $\delta \in (0,1)$ is the physical depreciation rate.[5] There is a given initial stock of capital $k_0 > 0$. We will write $i_t = \dot{k}_t$ for net investment. The representative agent has preferences over consumption paths represented by

$$\int_0^{\infty} e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt$$

where $\rho > 0$ is the subjective discount rate and $\sigma \geq 0$ the inverse of the intertemporal elasticity of substitution.

2.2 The relative price of equipment

Returns to scale differ between sectors. Since $\alpha < 1$, as the stock of capital grows, the equipment sector becomes more productive with respect to the consumption goods sector. This difference in productivity causes the decline of equipment prices relative to consumption goods prices. This difference in returns to scale can be interpreted in terms of the machines sector being more capital intensive than the consumption good sector or, as put forth by Boucekkine et al (2003), as a consequence of strong spillovers among equipment industries.[6]

[5]Observe that k_t is what in the literatute is referred to as effective or quality-adjusted productive capital. A number of authors discuss that quality-adjusted capital has to be constructed with the physical rather than the economic depreciation rate. This makes the assumption that δ is constant consistent with empirical studies. See the dicussion in section 3.3 in Cummins and Violante (2002) and the references therein.

[6]Cummins and Violante (2002) observe that their measure of investment-specific technical change occurs first in information technology and then accelerates in other industries. They conclude that information technology is a “general purpose” technology, an interpretation that matches well with the spillovers’ interpretation.

From the feasibility constraints, we can obtain the competitive equilibrium price of investment in terms of consumption units as the marginal rate of transformation:

$$q_t = -\frac{dc_t}{di_t} = -\frac{dc_t}{db_t}\frac{db_t}{di_t} = \frac{\alpha}{A} b_t^{\alpha-1}.$$

If the stock of machines used in the consumption good sector grows at a constant rate g, as it will be shown to be the case, the price of equipment decreases at rate $(\alpha-1)g < 0$. Note that this is a model of endogenous growth: there is no exogenous technical change determining this decline of the price of equipment.

2.3 Competitive equilibrium

In the absence of market failures, we can represent equilibrium allocations as solutions to the problem of a planner aiming at maximizing utility subject to the technological constraints. The Bellman equation associated to the planner's problem is

$$\rho v(k_t) = \max_{i=A(k-b)-\delta k} \frac{b^{\alpha(1-\sigma)}}{1-\sigma} + v'(k_t) i$$

where the constraint $c = b^\alpha$ has already been introduced in the objective function. Let μ_t be the Lagrange multiplier associated with the technological constraint. The first order conditions[7] are

$$\begin{aligned} \alpha b_t^{\alpha(1-\sigma)-1} - \mu_t A &= 0 \\ v'(k_t) - \mu_t &= 0 \end{aligned}$$

and from the envelope theorem

$$\rho v'(k_t) = v''(k_t) i_t + \mu_t (A - \delta).$$

Since $\mu_t = v'(k_t)$ we have $\dot{\mu}_t = v''(k_t)\dot{k}_t$. Then the envelope theorem equation reads $-\dot{\mu}_t/\mu_t = A - \delta - \rho$. Use the first equation to conclude that

$$-\frac{\dot{\mu}_t}{\mu_t} = A - \delta - \rho = -(\alpha(1-\sigma)-1)\frac{\dot{b}_t}{b_t}.$$

Denote $g = \dot{b}_t/b_t$ and solve for the growth rate of machines as

$$g = \frac{A-\delta-\rho}{1-\alpha(1-\sigma)}.$$

[7] This is a concave program. The first order conditions are sufficient if $\sigma \geq 1$. When $0 \leq \sigma < 1$, we have to require that $\rho > (1-\sigma)\alpha(A-\delta)$ so that we sufficiently future returns. Of course, in general, $A - \delta > 0$ is necessary for the problem to make sense, and $A - \delta > \rho$ for positive growth to be optimal. See Licandro and Felbermayr (2004) for the details.

From the feasibility constraints it is clear that g is the growth rate of capital, and therefore net investment i_t, and that αg is the growth rate of consumption. Observe that competitive equilibrium allocations are balanced growth paths as g is the growth rate of capital stock and investment for all t.

The competitive equilibrium allocation displays three regularities observed in actual data (see for example Whelan (2003)). Investment grows faster than consumption because $g > \alpha g$. The relative price of equipment decreases at rate $(\alpha - 1)g < 0$. Finally, the nominal share of investment in income remains constant. Indeed, from the equations of equilibrium, one can show after some calculations that

$$s_t = \frac{q_t i_t}{m_t} = \frac{q_t i_t}{c_t + q_t i_t} = \frac{\alpha(A - \delta - \rho)}{\rho(1-\alpha) + \alpha\sigma(A - \delta)} = s$$

for all t.

Let us take the consumption good as numeraire. Consequently, nominal income is $m_t = c_t + q_t i_t$. It is worth noting that this relation is not the resources constraint, since the transformation locus is

$$c_t = \left(k_t - \frac{i_t}{A}\right)^{\alpha}.$$

Measuring nominal income in consumption units is as arbitrary as measuring it in any other unit. In a later section, we will argue that Greenwood et al (1997) and Oulton (2002) identify production in the consumption good sector with output partly because of a resources constraint that makes this addition look a natural measure of output when in fact it is arbitrary. We also show that, from the perspective of the economic theory of index numbers, it provides with a poor measure of real output growth.

3 Actual measures of real growth

The model economy we just described is a world in which investment i_t grows faster than consumption c_t, the price of investment relative to consumption q_t declines, and the nominal shares of consumption and investment expenditures on income remain stable over time. That is, a reasonable replication of what we observe in actual data.

Departing from one-sector growth models raises the issue of aggregating consumption and investment growth to obtain a measure of real output growth. This is precisely what National Accounts have done switching from fixed-base to flexible-base quantity indexes. In this section we review these changes using the notation of our simple framework above.

3.1 Fixed-base quantity indexes

Traditional measures of real growth stem from or fixed-base (or fixed-weight) quantity indexes. The most common among these is the Laspeyres index.

Let us choose consumption as the numeraire so that its price is normalized to one while the price of investment in consumption units is q_t. The Laspeyres index fixes the price q_t of some base instant t and then computes the factor of change between t and $t+h$ as

$$\Pi^t_{t,t+h} = \frac{c_{t+h} + q_t i_{t+h}}{c_t + q_t i_t}$$

for all $h \geq 0$. The real growth rate measured by Laspeyres is the instantaneous growth rate of $\Pi^t_{t,t+h}$ as a function of h (see the appendix). That is,

$$\pi^t_{t,t+h} = \frac{d\Pi^t_{t,t+h}}{dh}\frac{1}{\Pi^t_{t,t+h}} = \frac{\dot{c}_{t+h} + q_t \dot{i}_{t+h}}{c_{t+h} + q_t i_{t+h}}.$$

The Laspeyres index is popular because it is conceptually simple and it provides with a natural concept of real GDP level if one considers only the numerator $c_{t+h} + q_t i_{t+h}$.

However, if the relative price of investment declines, so that real consumption and investment grow at different rates, the Laspeyres index tends to give too much weight to investment as we depart from the base year. In particular, if investment is growing faster than consumption, the Laspeyres growth rate tends to that of investment, therefore overstating real growth. Note that

$$\pi^t_{t,t+h} = \frac{\dot{c}_{t+h} + q_t \dot{i}_{t+h}}{c_{t+h} + q_t i_{t+h}} = \frac{c_{t+h}}{c_{t+h} + q_t i_{t+h}}\frac{\dot{c}_{t+h}}{c_{t+h}} + \frac{q_t i_{t+h}}{c_{t+h} + q_t i_{t+h}}\frac{\dot{i}_{t+h}}{i_{t+h}} \tag{1}$$

but the weight of consumption along an equilibrium path is

$$\frac{c_{t+h}}{c_{t+h} + q_t i_{t+h}} = \frac{1}{1 + q_t \dfrac{i_{t+h}}{c_{t+h}}} = \frac{1}{1 + \dfrac{q_t i_t}{c_t} e^{g(1-\alpha)h}},$$

and hence decreasing in h. This effect is known in the literature of index numbers as the substitution bias: the demand for goods whose price decline permanently usually display faster growth in real terms. Quantity indexes based on past (relatively high) prices overweight these items, overstating the real growth rate. The effect is larger the farther we are from the base year.

The Paasche index uses current prices as a base, and hence tends to understate real growth as we go back in time. The index is

$$\Pi^t_{t-h,t} = \frac{c_t + q_t i_t}{c_{t-h} + q_t i_{t-h}}$$

for all $h \geq 0$ and the growth rate measured by the index

$$\pi^t_{t-h,t} = \frac{d\Pi^t_{t-h,t}}{dh}\frac{1}{\Pi^t_{t-h,t}} = \frac{c_{t-h}}{c_{t-h} + q_t i_{t-h}}\frac{\dot{c}_{t-h}}{c_{t-h}} + \frac{q_t i_{t-h}}{c_{t-h} + q_t i_{t-h}}\frac{\dot{i}_{t-h}}{i_{t-h}}. \tag{2}$$

As h grows, so $t-h$ decreases, the weight of consumption increases because i_{t-h}/c_{t-h} decreases.

Fixed-base quantity indexes yield poor measures of real growth when GDP components grow at different rates because of changing relative prices.[8]

3.2 chained-type-type quantity indexes

The permanent decline in the price of equipment in actual data worsened when BEA started to correct computer prices by quality changes in the mid-1980's. The introduction of quality corrections in prices of equipment revealed a more acute declining pattern in the price of equipment relative to consumption goods. Consequently, real investment appeared to be growing even faster than real consumption. In this new scenario fixed-base quantity indexes faced the well-know substitution bias problem. National Accounts are moving to superlative indexes to measure real growth. In particular, fixed-base quantity indexes are today published together with a chained-type quantity index constructed from the Fisher Ideal index computed for every pair of years.

Chained-type indexes are used to filter price movements while updating the base therefore avoiding the substitution bias. National Accounts have chosen to use the Fisher Ideal index as a link because it is known to approximate some type of true quantity index. Between t and $t+h$ the Fisher Ideal index is the geometric mean of the Laspeyres index with base t and the Paasche index with base $t+h$, that is

$$F_{t,t+h} = \left(\Pi_{t,t+h}^{t} \Pi_{t,t+h}^{t+h}\right)^{\frac{1}{2}}.$$

The chained-type index is then computed recursively as

$$F_{0,t+h}^{C} = F_{0,t}^{C} F_{t,t+h}$$

each period nh, $n = 1, 2, ...$, starting from some date zero and setting $F_{0,0}^{C} = 1$ as the initial condition.[9] chained-type indexes can be constructed from other indexes. The Netherlands and Norway used to publish a chained-type index constructed from the Laspeyres index, but this chained-type index only has two out of the three interesting properties of the Fisher Ideal-based chained-type index:

[8]Updating regularly the base is not a solution because it goes against the very principle of fixed-base indexes. Further, publishing different series with different base-years would pose the problem of a multiplicity of measures of real growth for each period considered.

[9]Unlike fixed-base indexes, chain indexes do not have the multiplicative property. It is straightforward to check that, in general, the direct Fisher Ideal measurement $F_{0,2h}$ between zero and $2h$ does not coincide with the chain index that results from the product of the two intermediate Fisher Ideal indexes $F_{0,2h}^{C} = F_{0,h} F_{h,2h}$. Just observe that prices of instant h do not play any role in the calculation of $F_{0,2h}$. For this same reason, it is difficult to interpret a chain index as a real GDP series. These issues are very well illustrated in Whelan (2002).

- Computing the Laspeyres and Paasche indexes in fact filters price movements.
- Using prices of the current pair of periods updates the base continuously.
- Computing the average compensates the overstatement of the Laspeyres index with the understatement of the Paasche index, thus approximating a true quantity index.

Moreover, the Fisher Ideal index is a fair approximation of the Divisia index, and index built on a clear and appealing criterion. The Divisia index weights consumption and investment by the fraction the respective expenditures represent of total income. When the nominal shares change over time, the Divisia index takes the average of the nominal shares in t and $t+h$. In regard of expressions (1) and (2), as the period length h tends to zero, both indexes, and therefore the Fisher Ideal index, compute the same growth rate

$$d_t = \frac{c_t}{c_t + q_t i_t} \frac{c_t'}{c_t} + \frac{q_t i_t}{c_{t+h} + q_t i_t} \frac{i_t'}{i_t},$$

but this is precisely the definition of the Divisia index.[10] This equality is proved in more detail in the appendix.

4 Real growth in a two-sector AK model

The previous section shows that chained-type indexes yield conceptually better measures of real growth in a scenario in which consumption and investment grow at different rates and relative prices change. Further, in a model economy with explicit preferences, we can show that the Fisher Ideal index is in fact equal to the relevant true quantity index stemming from agents' preferences.

4.1 Instantaneous preferences

In regard of the Bellman equation above, the planner is choosing continuously between consumption and investment. Hence, we can interpret

$$w_t(c,i) = \frac{c^{1-\sigma}}{1-\sigma} + v'(k_t)i$$

to represent instantaneous preferences over contemporaneous consumption and investment. To save notation, we write $w_t(c,i)$ rather than $w(c,i,k_t)$ but time enters this

[10] In continuous-time, since the Laspeyres and Paasche index tend to each other as h goes to zero, the chain indexes based on the Laspeyres and the Fisher Ideal indexes coincide, and both are therefore equal to the Divisia index. In discrete-time, however, only the Fisher Ideal index approximates the Divisia index.

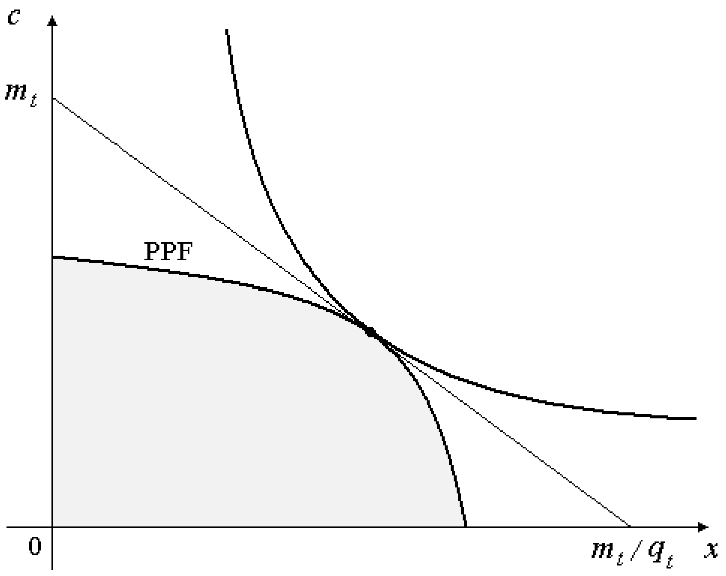

Figure 1: The production possibilities frontier and competitive prices

function only through the stock of capital k_t.[11] The same comment applies to the indirect utility and expenditure functions below. Recall that nominal income was defined as $m_t = c_t + q_t i_t$. Figure 1 depicts the budget constraint jointly with the production possibilities frontier and an indifference curve. The constraints in the maximization problem of the Bellman equation can be replaced by the budget constraint $c + q_t i \leq m_t$ because the budget line is tangent to the production possibilities frontier locally at the optimum.

4.2 The Fisher-Shell true quantity index

Since the utility function that summarizes preferences in instant t is time dependent, the natural choice to measure real growth is the true quantity index introduced by Fisher and Shell (1971). Define the indirect utility function as

$$u_t(m_t, q_t) = \max_{c+q_t i \leq m_t} w_t(c, i)$$

and the associate expenditure function

$$e_t(u_t, q_t) = \min_{w_t(c,i) \geq u_t} c + q_t i.$$

Since utility comparisons must be done using the same preference map, the Fisher-Shell true quantity index fixes both prices and preferences. In particular, it compares income

[11] The planner of our economy solves a standar recursive program in which the state variable summarizes at each instant all information from the past that could be relevant for the choice today. For a brief exposition of recursive techniques in continuous-time see Obstfeld (1992).

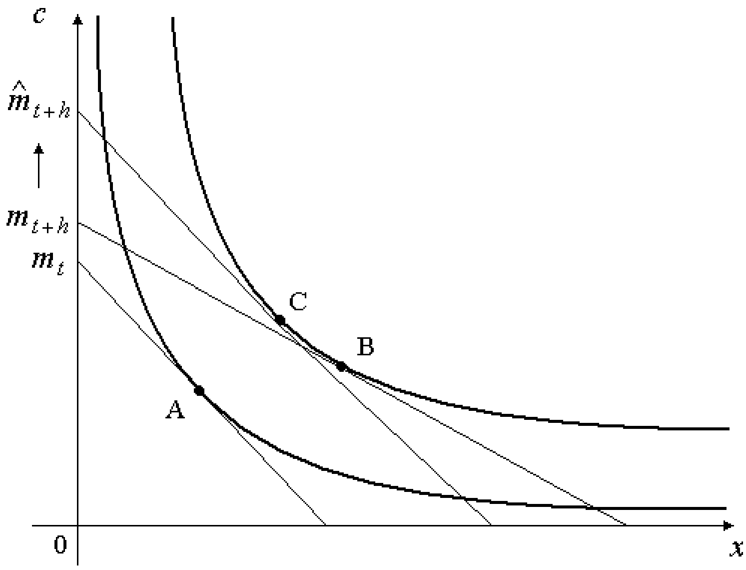

Figure 2: The Fisher-Shell true quantity index

today m_t with the hypothetical level of income $\hat{m}_{t+h}$ that would be necessary to attain the level of utility associated with tomorrow's income and prices m_{t+h}, q_{t+h} with today's prices q_t and today's preferences as evaluated by e_t, u_t. Denote this artificial level of income as

$$\hat{m}_{t+h} = e_t[u_t(m_{t+h}, q_{t+h}), q_t].$$

The procedure is depicted in Figure 2. The preference map corresponds to preferences of instant t as represented by w_t. Point A is the current situation in instant t. Point B is the choice of instant t preferences when we face instant $t+h$ prices q_{t+h} and income m_{t+h}. Point C represents the choice that maintains such level of utility but with instant t prices q_t. In the end, we compare two levels of income that correspond to the same price vector so it is clear that we are extracting price changes. In this particular case, the true quantity index is just reflecting the fact that the true output deflator is dropping with the price of equipment, that is to say that income in real terms is growing more than m_{t+h}/m_t.

The instantaneous Fisher-Shell real growth rate is

$$f_t = \frac{d}{dh} \left. \frac{\hat{m}_{t+h}}{m_t} \right|_{h=0} = \frac{1}{m_t} \left. \frac{d\hat{m}_{t+h}}{dh} \right|_{h=0}, \tag{3}$$

that is, the instantaneous growth rate of the factor defined above as h gets small.[12] To

[12] In continuous-time, it does not make a difference whether we define the true quantity index like we do or in terms of $m_t/\hat{m}_{t-h}$. See again the appendix for a rationale of this definition.

compute this index note that

$$\left.\frac{d\hat{m}_{t+h}}{dh}\right|_{h=0} = e_{1,t}[u_t(m_t, q_t), q_t](u_{1,t}(m_t, q_t)\dot{m}_t + u_{2,t}(m_t, q_t)\dot{q}_t)$$

where the subscript denotes the partial derivative with respect to the corresponding argument. To obtain an expression for all these derivatives let us go back to the definitions above. Let μ be the Lagrange multiplier of the maximization problem in the definition of the indirect utility function. We have, from the first order conditions, that $w_{1,t}(c_t, i_t) = \mu$, so that

$$\begin{aligned}\frac{du_t(m_t, q_t)}{dm_t} &= \mu = w_{1,t}(c_t, i_t) \\ \frac{du_t(m_t, q_t)}{dq_t} &= -\mu i_t = -w_{1,t}(c_t, i_t)i_t.\end{aligned}$$

Let λ be the Lagrange multiplier of the minimization problem in the definition of the expenditure function. From the first order conditions $w_{1,t}(c_t, i_t) = -\lambda^{-1}$, and hence

$$\frac{de_t(u_t, q_t)}{du_t} = -\lambda = \frac{1}{w_{1,t}(c_t, i_t)}.$$

We conclude that

$$f_t = \frac{1}{m_t}\frac{1}{w_{1,t}(c_t, i_t)}\left(w_{1,t}(c_t, i_t)\dot{m}_t - w_{1,t}(c_t, i_t)i_t\dot{q}_t\right) = \frac{\dot{m}_t - i_t q\dot{q}_t}{m_t} = \frac{\dot{m}_t}{m_t} - \frac{q_t i_t}{m_t}\frac{\dot{q}_t}{q_t}.$$

Differentiate $m_t = c_t + q_t i_t$ with respect to time and note that $q_t i_t / m_t = s$ to write

$$\frac{\dot{m}_t}{m_t} = (1-s)\frac{\dot{c}_t}{c_t} + s\frac{\dot{i}_t}{i_t} + s\frac{\dot{q}_t}{q_t}$$

and then

$$f_t = \frac{\dot{m}_t}{m_t} - s\frac{\dot{q}_t}{q_t} = (1-s)\frac{\dot{c}_t}{c_t} + s\frac{\dot{i}_t}{i_t} = (1-s)\alpha g + sg$$

for all t, but this is the expression of the Divisia index so that $f_t = d$ for all t.[13]

The interpretation is straightforward. It is clear that f is a measure of real growth since it is constructed as the growth rate of nominal income substracting pure price changes, in this case the change of the relative price of investment q_t. The index only keeps changes in quantities. It is also clear that it is a true index because it is constructed from the representative agent's preferences.[14]

[13]This equivalence would come as no surprise to index number theorists. The Fisher Ideal index is know to approximate in general some sort of true quantity index because both are bounded from above and below by the Laspeyres and Paasche indexes respectively. In continuous-time, these indexes tend to each other as the time interval h tends to zero. Further, in general, the Divisia index coincides with the Fisher Ideal index if the growth rates of consumption and investment are constant.

[14]At this point it may be worth clarifying that it is not GDP but NDP what matters for

5 Discussion

In principle, it is very appealing, if current and future consumption is all that matters to utility, that the growth rate of consumption should summarize all what the agent cares about. In this section we will give an interpretation for the growth rate of investment playing a role in the true quantity index. Further, in the stylized context of the AK model, we will aim at clarifying one or two points that are still under debate in the growth theory literature.

5.1 Investment also matters

One way to see that the investment growth rate reflects in part welfare is to note that the Fisher-Shell-Divisia index can be expressed as

$$d = (A - \delta)s.$$

The index is therefore reflecting that potential maximum growth $A - \delta$ will be exploited to the extent that agents care for future consumption as represented by the propensity to invest s. But this is to say that the investment growth rate, reflecting the concern for future consumption, conveys information that is relevant for welfare.

This interpretation can be further supported by a simple example. The basic intuition is that the same growth rate for consumption can stem from different parameters' profiles that, eventually, reflect different preferences. Consider the economy described above and a second economy such that $\rho > \tilde{\rho}$ and $\delta < \tilde{\delta}$. Suppose further that $\delta + \rho = \tilde{\delta} + \tilde{\rho}$. The rest of parameters are common to both economies. In regard of the equilibrium expressions for s and g, we have $g = \tilde{g}$ and therefore $\alpha g = \alpha \tilde{g}$ so that consumption grows at the same rate in both economies. Since optimal paths are balanced growth paths, we can easily derive the functional form of the value function associated to the planner's problem

$$v(k) = \frac{\theta^{\alpha(1-\sigma)-1}}{A} \frac{k^{\alpha(1-\sigma)}}{1-\sigma} \tag{4}$$

where $\theta = b_t/k_t$ can be expressed in terms of parameters as

$$\theta = \frac{\rho - \alpha(1-\sigma)(A-\delta)}{A(1-\alpha(1-\sigma))}.$$

Hence, decreasing ρ in the same amount as δ increases the numerator because $-1 + \alpha(1 - \sigma) < 0$, while this same expression, negative, is the power of θ in (4).

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welfare, and in fact it is the contribution of net investment i_t what the true quantity index is taking into account. The fraction of k_t that is interpreted to depreciate physically is a lost resource that does not contribute to welfare. It is in this sense that some authors claim that NDP is relevant for welfare and GDP for productivity (see the discussion in Oulton (2002)).

In short, the growth rate of consumption is not a good measure of real growth because it is unable to reflect the welfare differences between these two economies: the second economy weights more future consumption so that values more than the first the same growth rate of consumption. The Fisher-Shell-Divisia index does reflect the differential in welfare. Since $g = \tilde{g}$, if there is a difference between d and $\tilde{d}$ it has to be because of differences in the propensity to invest. Check that

$$s = \frac{\alpha(A - \delta - \rho)}{\rho(1-\alpha) + \alpha\sigma(A - \delta)} = \frac{\alpha(A - \tilde{\delta} - \tilde{\rho})}{\rho(1-\alpha) + \alpha\sigma(A - \delta)} < \frac{\alpha(A - \tilde{\delta} - \tilde{\rho})}{\tilde{\rho}(1-\alpha) + \alpha\sigma(A - \tilde{\delta})} = \tilde{s}$$

where the equality follows from the imposition that $\delta + \rho = \tilde{\delta} + \tilde{\rho}$ and the inequality from rising δ and decreasing ρ in the denominator. But then

$$d = (\alpha + s(1-\alpha))g < (\alpha + \tilde{s}(1-\alpha))g = \tilde{d}$$

so that $d < \tilde{d}$ is reflecting the higher valuation in the second economy.

Quantity indexes reflect imperfectly welfare but this example shows that the Divisia index is a better representation of preference than the consumption growth rate.

5.2 It is not the choice of the numeraire

In order to measure real growth, Hulten (1992, equation (7)) follows Jorgenson (1966) and suggests a row addition of consumption and investment units, calling the outcome $c_t + i_t$ quality-adjusted output. Greenwood et al (1997) note that, in their setting, adding consumption and effective investment turns the economy into a standard Solow (1960) growth model with no embodied technical change.[15] In regard of the standard general equilibrium concept of commodity, they correctly state that output in the consumption good sector requires investment to be deflated by its relative price, in our notation $c_t + q_t i_t = m_t$. However, they go on identifying nominal income m_t in consumption units with real output so that their measurement of real GDP growth is the consumption growth rate.[16]

Doing so, Greenwood et al (1997) implicitly adhere to a position often taken in the growth accounting literature. In a model with linear preferences, Weitzman (1976) discussed that net investment has to be added to consumption if we wanted to have a measure of the level of income that was welfare-relevant. A secondary point in his reasoning is the statement that nominal income has to be measured in consumption units. This choice

[15] See Hercowitz (1998) for a review of the Solow-Jorgenson controversy.

[16] In their setting this choice looks somewhat natural because the machines' sector uses as input the consumption good so that the equivalent of $m_t = c_t + q_t i_t$ is total output in the non-durables sector.

seems natural if we are measuring levels, and nevertheless arbitrary. Oulton (2004) generalizes the argument to concave preferences but then translates the argument to growth rates. In practice, he suggests that GDP components have to be deflated by the consumption price index in order to measure growth. In terms of the notation above, he suggest to use $m_t = c_t + q_t i_t$ to measure the level and the real growth rate, but m_t grows at the same rate as consumption $\alpha g < d$. This is in effect what Greenwood et al (1997) suggest when they identify income m_t with real output.

The argument is appealing because, in principle, consumption is all that matters for utility. However, it is clear that the measurement of real growth is not a matter of numeraire choice. Any reasonable measure of real growth has to be between these two extremes and should not depend on the units one chooses to express nominal income. One can check that, indeed, the Fisher-Shell true quantity is independent of the choice of the numeraire. Consider any price paths for consumption $p_{c,t}$ and investment $p_{i,t}$, but imposing $p_{i,t}/p_{c,t} = q_t$, where q_t is our equilibrium price of equipment in consumption units. Define nominal income as $n_t = p_{c,t}c_t + p_{i,t}i_t$.Then of course $n_t = p_{c,t}m_t$ but it is also straightforward to check that $\hat{n}_{t+h} = p_{c,t}\hat{m}_{t+h}$. But then the quantity index is $\hat{n}_{t+h}/n_t = \hat{m}_{t+h}/m_t$, it does not depend on the price normalization we choose. In short, the debate on the measurement of real growth should not be on the choice of a numeraire but of a reasonable quantity index.

5.3 Contributions to growth

In this paper we have discussed that the true quantity index is the only internally consistent way to measure real growth in a model economy. But then, we have to view model measures with the same precaution as actual data. For instance, the concept of real share is, in general, at least elusive (see section 4 in Whelan (2002)). In our model economy, the consumption growth rate αg is smaller than the investment growth rate g so that any reasonable measure of real growth d has to be in between. Then, of course, $d < g$, but it is dubious that we can conclude that the real share of investment in GDP is growing. The difficulty stems from the fact that we lack a concept of real GDP, at least not a clear as a concept as it was the case with the Laspeyres aggregate.

This is also important from the viewpoint of actual data regularities. One may wonder whether it makes sense to use as a regularity something like the share of investment in real GDP if we do not have a satisfactory concept of real GDP either in actual data or within the model.

Appendix

A Quantity indexes in continuous-time

Suppose we have a general definition of an index $\Gamma_{t,t+h}$ interpreted as a gross rate (or factor) of growth between t and $t+h$. We can define the instantaneous growth rate of the index in instant t as

$$\gamma_t = \left.\frac{d\Gamma_{t,t+h}}{dh}\frac{1}{\Gamma_{t,t+h}}\right|_{h=0}. \tag{5}$$

The intuition of (5) is clear if one observes that the growth rate of a factor of change of a continuous-time variable is equal to the growth rate of the variable itself. Let x_t be a continuous-time variable and fix some reference point in time t. The growth rate in instant $t+h$ can be seen as the growth rate of $\Gamma_{t,t+h} = x_{t+h}/x_t$ because

$$\gamma_{t,t+h} = \frac{d\Gamma_{t,t+h}}{dh}\frac{1}{\Gamma_{t,t+h}} = \frac{x'_{t+h}x_t}{x_t^2}\frac{1}{\frac{x_{t+h}}{x_t}} = \frac{x'_{t+h}}{x_{t+h}}$$

and of course the growth rate in instant t is just

$$\gamma_t = \left.\frac{d\Gamma_{t,t+h}}{dh}\frac{1}{\Gamma_{t,t+h}}\right|_{h=0} = \left.\frac{x'_{t+h}}{x_{t+h}}\right|_{h=0} = \frac{x'_t}{x_t}.$$

This way of defining the instantaneous growth rate may look odd but it may be useful in those cases in which we have an index like $\Gamma_{t,t+h}$ but no clear variable giving rise to this index like x_t in this example. The Fisher Ideal index is one of these cases.

Using the notation introduced in section 2, the starting point is some nominal aggregate $c_t + q_t i_t$. Fixed-base quantity indexes use instant ℓ prices as weights and compute an index that is equal to a factor of growth

$$\Pi_t^\ell = \frac{c_t + q_\ell i_t}{c_\ell + q_\ell i_\ell}.$$

When we measure real growth over an interval $[0, T]$ the index Π_t^ℓ is the Laspeyres index when $\ell = 0$ and the Paasche index when $\ell = T$. The Fisher Ideal index between t and $t+h$ is defined as

$$F_{t,t+h} = \left(\Pi_{t,t+h}^t \Pi_{t,t+h}^{t+h}\right)^{\frac{1}{2}}. \tag{6}$$

There is a clear parallelism between $F_{t,t+h}$ and $\Gamma_{t,t+h}$ but there is no counterpart for the x_t variable above. Note that the definition of the index itself requires to set some reference point in time t and a second reference $t+h$. Further, the non-linearity of expression (6) does not make it easy to turn some discrete-version of the growth rate into a derivative that could yield an instantaneous growth rate in an intuitive way.

Instead, we can apply the definition (5) above and define

$$g_t = \left.\frac{dF_{t,t+h}}{dh}\frac{1}{F_{t,t+h}}\right|_{h=0}.$$

With this definition at hand, it is straightforward to check that the continuous-time equivalent of the Fisher Ideal index is in fact equal to the Divisia index. We have

$$\begin{aligned}\frac{dF_{t,t+h}}{dh}\frac{1}{F_{t,t+h}} &= \frac{1}{2}\left(\Pi^t_{t,t+h}\Pi^{t+h}_{t,t+h}\right)^{-1}\left(\frac{d\Pi^t_{t,t+h}}{dh}\Pi^{t+h}_{t,t+h}+\Pi^t_{t,t+h}\frac{d\Pi^{t+h}_{t,t+h}}{dh}\right)\\ &= \frac{1}{2}\left(\frac{d\Pi^t_{t,t+h}}{dh}\frac{1}{\Pi^t_{t,t+h}}+\frac{d\Pi^{t+h}_{t,t+h}}{dh}\frac{1}{\Pi^{t+h}_{t,t+h}}\right).\end{aligned}$$

Then note that

$$\left.\frac{d\Pi^t_{t,t+h}}{dh}\frac{1}{\Pi^t_{t,t+h}}\right|_{h=0} = \left.\frac{c'_{t+h}+q_t i'_{t+h}}{c_t+q_t i_t}\right|_{h=0} = \frac{c'_t+q_t i'_t}{c_t+q_t i_t}$$

while

$$\begin{aligned}&\left.\frac{d\Pi^{t+h}_{t,t+h}}{dh}\frac{1}{\Pi^{t+h}_{t,t+h}}\right|_{h=0}\\ &\quad= \left.\frac{\left(c'_{t+h}+q'_{t+h}i_{t+h}+q_{t+h}i'_{t+h}\right)\left(c_t+q_{t+h}i_t\right)-\left(c_{t+h}+q_{t+h}i_{t+h}\right)q'_{t+h}i_t}{\left(c_t+q_{t+h}i_t\right)^2}\right|_{h=0}\end{aligned}$$

and therefore

$$\left.\frac{d\Pi^{t+h}_{t,t+h}}{dh}\frac{1}{\Pi^{t+h}_{t,t+h}}\right|_{h=0} = \frac{c'_t+q'_t i_t+q_t i'_t-q'_t i_t}{c_t+q_t i_t} = \frac{c'_t+q_t i'_t}{c_t+q_t i_t}.$$

We conclude that

$$\begin{aligned}g_t = \left.\frac{dF_{t,t+h}}{dh}\frac{1}{F_{t,t+h}}\right|_{h=0} = \frac{1}{2}\left(\frac{c'_t+q_t i'_t}{c_t+q_t i_t}+\frac{c'_t+q_t i'_t}{c_t+q_t i_t}\right) &= \frac{c'_t+q_t i'_t}{c_t+q_t i_t}\\ &= \frac{c_t}{c_t+q_t i_t}\frac{c'_t}{c_t}+\frac{q_t i_t}{c_t+q_t i_t}\frac{i'_t}{i_t},\end{aligned}$$

that is, the Divisia index.

The definition above (5) is also useful applied to the Fisher-Shell quantity index since we have a well-defined index $\hat{m}_{t+h}/m_t$ but it is not clear who would play the role of x_t in this case. Section 4 takes this viewpoint and defines the growth rate as in expression (3) in page 10.

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