Changing Income Inequality and Panel Income Changes in Times of Economic Growth and Economic Decline

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Abstract

When economic growth (or economic decline) takes place, who benefits and who is hurt how much? The more traditional way of answering this question is to compare two or more comparable cross sections and gauge changing income inequality among countries or individuals. A newer way is to utilize data on a panel of countries or a panel of people and assess the pattern of panel income changes. How do these two approaches relate to one another? This paper shows, first, that it is possible to have all four combinations - rising or falling inequality and divergent or convergent panel income changes, and second, under what conditions, for various measures of rising/falling inequality and various measures of divergent/convergent income changes, each of the four possible combinations can arise.

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1 Introduction

Recently, one of the authors of this paper had the privilege of meeting with a Nobel Prize-winning economist who had just published a major new book on economic inequality. The Nobel Laureate had presented extensive evidence documenting the rise of income inequality in the United States. The author said to the Nobel Laureate, “Yes, but in our research, we found that when we follow the same people over time, those who earned the least to begin with gained more in dollars than those who started at the top of the earnings distribution.” Said the Nobel Laureate: “That cannot be - the income distribution would have had to have become more equal.”

The literature also offers a claim regarding the opposite set of circumstances. Consider a panel of countries with per capita incomes in comparable currency units - Purchasing Power Parity-adjusted dollars, for example. Define β-divergence (convergence) as arising when a regression of final log-income on initial log-income produces a regression coefficient greater than (less than) one. Define σ-convergence (divergence) as arising when the variance of log-incomes falls (rises) from the initial year to the final year. It is proven in the literature that β-divergence measured in this way and σ-convergence measured in this way cannot arise simultaneously - more specifically, σ-convergence implies β-convergence but β-convergence does not imply σ-convergence (Furceri, 2005; Wodon and Yitzhaki, 2006).

Is it possible to have convergent panel income changes, that is, the income changes we see following named individuals over time, and rising income inequality? Is it possible to have divergent panel income changes and falling income inequality? Are the possibilities in times of economic growth different from those in times of economic decline? One purpose of this paper is to derive what is possible and what is impossible. Contrary to the suggestions in the two preceding paragraphs, we show that it is indeed possible to have rising or falling inequality along with convergent or divergent mobility, both in times of economic growth and in times of economic decline (see Table 1).

The second purpose of this paper is to derive conditions under which, for various measures of rising/falling inequality and various measures of convergent/divergent income changes, each of the four possibilities can arise. A number of propositions are derived.

Overall, the results in this paper reaffirm what has been known in the literature for some time. Whether income inequality rises or falls in the cross section is one thing. Whether panel income changes are divergent or convergent is another thing. Rising/falling inequality and divergent/convergent income changes are both interesting; they are, however, different.

But the results here are not just a reaffirmation. This paper goes beyond the previous literature in deriving precise conditions under which i) income inequality rises or falls, ii) panel income changes are divergent or convergent, iii) the four possibilities in Table 1 can arise, and iv) when certain combinations cannot arise for particular measures of changing inequality and convergence/divergence.

2 Measurement Issues and Links to the Literature

The two key variables in this research are income inequality and panel income changes. "Income" is the term used for the economic variable of interest, which could be total income, labor earnings, consumption, or something else. The income recipient will be called a "person", but the results apply equally to households, workers, per capitas, or adult equivalents.

2.1 Income Inequality

When is income inequality rising or falling? This question is answered by using a functional or an index to represent the inequality at one point in time and at another point in time and then to compare them.

Income inequality and the change in income inequality are conceptualized and measured in a number of ways. "Relative inequality" is concerned with income comparisons measured in terms of ratios, "absolute inequality" with income comparisons measured in terms of dollar differences.

A widely-used criterion for determining which of two income distributions is relatively more equal than another is the three-part Lorenz criterion, which states i) if Lorenz curve A lies somewhere above and never below Lorenz curve B, A is more equal than B, ii) if Lorenz curves A and B coincide, then A and B are equally unequal, and iii) if the Lorenz curves of A and B cross, the relative inequalities of A and B cannot be compared using the Lorenz criterion alone. Judging a Lorenz-dominant distribution to be more equal than a Lorenz-dominated one is equivalent to making inequality comparisons on the basis of four commonly-accepted relative inequality axioms: anonymity, scale-independence, population-independence, and the transfer principle (Fields and Fei, 1978).

Yet, despite its appeal, the Lorenz criterion is not universally used for two reasons: its ordinality and its incompleteness. When the Lorenz criterion does render a verdict about which of two income distributions is more equal than another, it can only say that A is more equal than B but not how much more equal A is than B. And when Lorenz curves cross, the Lorenz criterion cannot render a verdict.

Those analysts who seek a complete cardinal comparison of the inequalities of two income distributions are led to use one or more inequality indices. For present purposes, these indices can be put into three categories:

1. Lorenz-consistent relative inequality indices: An inequality index is Lorenz-consistent if, when one Lorenz curve dominates another, the index registers the dominant distribution as (weakly) more equal. A partial listing of Lorenz-consistent relative inequality indices includes the Gini coefficient, income share of the richest X%, income share of the poorest Y%, Atkinson index, Theil index, and the coefficient of variation and its square. For details, see Sen (1997) and Cowell (2011).

2. Lorenz-inconsistent relative inequality indices: An inequality index is Lorenz-inconsistent if, when one Lorenz curve dominates another, it is ever the case that the index shows the Lorenz-dominant distribution to be less equal. One commonly-used relative inequality index is Lorenz-inconsistent: the variance of the logarithms of income. This index violates the transfer principle - that is, it is possible to make a rank-preserving transfer of income from a relatively rich person to a relative poorer person and yet the index can register an increase in relative inequality (Foster and Ok, 1999; Cowell, 2011).

3. Lorenz-inconsistent absolute inequality indices: All absolute inequality indices are Lorenz-inconsistent because they violate the axiom of scale-independence. For example, the variance of incomes is not Lorenz-consistent: doubling everyone's income increases inequality by a factor of four.

In our work below, we emphasize Lorenz curve comparisons and Lorenz-consistent inequality indices. However, we also give attention to the variance of incomes and the variance of log-incomes despite their Lorenz-inconsistency, in part because the literature has done so and in part because results can be gotten using them. These various ways of measuring inequality are summarized in Part A of Table 2.

2.2 Divergent and Convergent Panel Income Changes

By definition, income mobility analysis entails looking at the joint distribution of incomes at two or more points in time. This is an analysis of panel income changes since we follow a particular individual. Our analysis in this paper is limited to income changes between an initial period and a final period.

The income mobility literature distinguishes six mobility concepts: time-independence, positional movement, share movement, directional income movement, non-directional income movement, and mobility as an equalizer of longer-term incomes relative to initial (Fields, 2008). For purposes of characterizing the pattern of panel income changes, the relevant concept is directional income movement among panel people - that is, who gains or loses how much, from an initial date to a final one.

Panel income changes are said to be divergent when the income recipients who started ahead on average get ahead faster than those who started behind. It is convergent when those who started ahead on average get ahead more slowly than those who started behind. It is neutral when neither is the case.

What it means to get ahead at a faster, slower, or same rate itself requires careful specification. In the macroeconomics literature, the object of interest is nearly always the growth rate in percentages, often approximated by changes in log-income (see, for example, Barro, 1991; Sala-i-Martin, 1996). On the other hand, the literature on panel income changes among individuals or households presents a more varied picture; some studies use income changes in dollars, while others use changes in log-dollars, percentage changes, changes in income shares, or changes in income quantiles such as deciles or centiles (Jäntti and Jenkins, 2013).

Much of the literature assesses divergence or convergence by assuming a linear relationship between final income and initial income or between income change and initial income. Much but not all of our analysis works with the linearity assumption as well.

Accordingly, we gauge divergence or convergence as follows. Consider a generic income variable y, which might be dollars, log-dollars, or income shares. We can have the levels-on-levels regression $y_1 = \alpha_y + \beta_y y_0 + u_y$ or the change-on-initial regression $\Delta y = \gamma_y + \delta_y y_0 + u_y$. The two regressions are linked by the relationship $\delta_y = \beta_y - 1$. Divergence is said to arise when $\beta_y > 1$, or equivalently, when $\delta_y > 0$. Likewise, we have convergence when $\beta_y < 1 \iff \delta_y < 0$. We may be interested in divergence or convergence of true proportional changes (as opposed to the logarithmic approximation thereof), in which case we would want to regress the proportional change in dollars on initial dollars: pch d $\equiv (d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}$. Proportional changes are divergent or convergent as θ is greater or less than zero.

These various approaches to divergent and convergent panel income changes are summarized in Part B of Table 2.

2.3 Links to the Empirical Literature

The empirical macroeconomics literature on rising/falling inequality and divergent/convergent income changes across countries is highlighted by the pathbreaking papers of Barro (1991), Sala-i-Martin (1996), and Quah (1993); for empirical reviews, see Durlauf and Quah (1999) and Korotayev *et al.* (2011). For the most part, the literature reports σ-convergence (which signifies falling inequality of log-incomes across countries) and β-convergence (which signifies countries moving closer together over time in mean log-incomes). See, however, the paper by Pritchett (1997), which carries the provocative title "Divergence, Big Time".

The microeconomics literature on rising/falling inequality and divergent/ convergent panel income changes across individuals within a country reveals a pronounced pattern: generally, those income recipients with the lowest reported incomes or earnings to begin with are the ones that experienced the most positive or least negative panel income changes both in percentages (or the log-dollar approximation thereto) and in dollars, regardless of whether income inequality in their countries was rising or falling. Empirically, unconditional convergence in percentages has been found in studies of panel income changes in Indonesia, Venezuela, Tanzania, South Africa, and China for the period 1991-1995. Unconditional convergence in dollars has been found in studies of panel income changes in Indonesia, South Africa, Spain, Venezuela, South Africa, Argentina, Mexico, the United States, Nicaragua, the Philippines, and Albania. The one important instance in which unconditional divergence is reported is China for the period 1998-2002 (Ying *et al.*, 2006). For references, see Fields (2010) and the other chapters in World Bank (2010).

In sum, in both the macroeconomics literature and the panel income changes literature, convergence is the rule.

3 Possibility Results

3.1 A Multi-Person Example

Let A $\rightarrow$ B be a change in incomes from initial year to final year. This change can be analyzed in anonymous terms or in panel terms.

The (usual) anonymous analysis arrays income recipients from lowest income to highest in each year. We will denote anonymous distributions using ( ). For example, let the initial cross section distribution be (1, 2, 2, 2, 3) and the final cross section distribution be (1, 2, 2, 2, 5). Clearly, economic growth has taken place, all of the economic growth took place for the richest anonymous quintile, and incomes were unchanged in all other quintiles.

Now let us take a panel data approach. Adopt the notational convention of arraying individuals in increasing order of incomes in the base year and keeping them in the same place in the vector in the final year. We will denote panel distributions using square brackets [ ]. Suppose in the preceding example A $\rightarrow$ B is [1, 2, 2, 2, 3] $\rightarrow$ [5, 2, 2, 2, 1]. When we follow the same people over time, the richest person loses, the poorest person gains, and the middle income persons' incomes are unchanged. These panel income changes are convergent, measured linearly or otherwise. Yet, despite these changes being convergent, everything about the anonymous changes in the previous paragraph remains true - in particular, rising inequality.

This example illustrates that rising inequality and convergent panel income changes are mutually consistent. Conditions for the two to arise simultaneously are developed below.

3.2 A Two Person Example: Patterns of Inequality Change and Divergence/Convergence

To help us understand the relationship between rising and falling inequality on the one hand and divergent/convergent income changes on the other, the following two-person example has proven instructive. Please refer to Table 3.

Suppose we have two persons with initial incomes \$5 and \$20 respectively. Let the economy experience 20% economic growth, so that total income increases from \$25 to \$30. Consider all possible values of income changes for the two individuals ranging from the initially-poorest person losing everything to the initially-poorest person gaining everything. The initial incomes, final incomes, and income changes for the two persons 1 and 2 are displayed in the left third of the table.

The middle third of the table displays the change in relative inequality for each possible pattern of gains and losses.

Let us look first at inequality change as gauged by Lorenz-worsening (which signifies rising inequality), Lorenz-coincidence (which signifies constant inequality), and Lorenz-improvement (which signifies falling inequality). We observe five ranges for these ordinal changes: Lorenz-worsening, followed by Lorenz-coincidence, followed by Lorenz-improvement, followed by Lorenz-coincidence, followed by Lorenz-worsening.

We then measure the change in inequality cardinally using three inequality indices: the change in the coefficient of variation squared (which is a Lorenz-consistent relative inequality index), the change in the variance of incomes in log-dollars (which, though a relative inequality index, is not Lorenz-consistent), and the change in the variance of incomes in dollars (which is an absolute inequality index, not a relative one). The changes using the two relative inequality indices follow identical patterns, though with different numerical values:

- First, they decrease together.
- Then they hit zero together. This occurs at [6, 24], which has the same relative inequality as the pre-growth distribution [5, 20].
- Then they continue to decrease together until [15, 15], which they should, because this is a perfectly equal distribution of income.
- Then they rise together until they hit zero at [24, 6], which also has the same relative inequality as the pre-growth distribution [5, 20].
- Then they continue to rise together.

As for absolute inequality as measured by the variance of incomes in dollars, it too follows an inverted-U pattern, reaching its minimum at [15, 15]. Note, though, that the variance crosses zero at a different point from the Lorenz curves and the two relative inequality indices. This is to be expected: relative inequality is unchanged when economic growth has been proportionate, but absolute inequality increases.

Note one other aspect about the inequality section of the table: it is entirely symmetric around the perfect equality point (15, 15). This is exactly as it should be: income inequality is unchanged when incomes are permuted across individuals.

Finally, let us look at the panel income changes. There are two points to be made.

First, the panel income changes proceed monotonically from most divergent to most convergent, however they are measured. This is exactly what would be expected as the initially-poorest person gains more and more and more, and the initially-richest person gains less and less and less.

And second, the regressions of final share on initial share, final log-dollars on initial log-dollars, and proportional changes on initial income all cross the divergence/convergence cutoff (1.00 for the levels regressions, 0 for the proportional change regression) at the exact same point. However, the dollars-on-dollars regression goes from divergent to convergent later - to be specific, at the precise point where each person gains the same amount, which in this case is +2.5 for each individual. So if convergence in dollars is found in periods of economic growth, which often it is empirically, it is a stronger result than a finding of convergence in shares, log-dollars, or proportional changes.

3.3 A Matrix of Possibilities

In what came before, we identified four ways of determining the direction of change in relative inequality - i) Lorenz-improvement and Lorenz-worsening, ii) Change in a Lorenz-consistent relative inequality index such as the coefficient of variation squared or the income share of the poorest quantile, iii) Change in the variance of log-dollars, which is a Lorenz-inconsistent relative inequality measure, and iv) Change in the variance of dollars, which is an absolute inequality measure - and four ways of assessing divergence or convergence: i) Dollar changes, ii) Share changes, iii) Log-dollar changes, iv) Proportional changes.

Is it possible to have each possible combination of rising or falling relative inequality and divergent or convergent panel income changes? The answer is yes, provided they are measured suitably. Table 4 displays examples for each of the thirty-two cells.

Most of the examples come directly from Table 3. They illustrate that just two people are needed for most of the combinations. But to get the remaining combinations, we need to complicate the examples by adding more people and choosing our measures carefully. The next subsection provides details.

3.4 A Further Look at Divergent Income Changes and Falling Inequality

As noted in the introduction, Furceri (2005) and Wodon and Yitzhaki (2006) proved the impossibility of divergent log-dollar changes and falling relative inequality as measured by variance of log-dollars. But the examples in Table 4 show that it is possible to have divergent panel changes and falling relative inequality simultaneously when other measures are used. Let us look at this further.

Dealing first with the measurement of divergence, let us explore divergence in dollars rather than in log-dollars. The consistency of divergent dollar changes and falling variance of log-dollars can be proved by example. In a two-person economy, dollar changes are divergent if the richer person gains more dollars than the poorer person. Also, in a two-person economy, if both persons gain and the richer person gains a smaller percentage of income than does the poorer person, the variance of log-dollars falls. An example is [5, 20]→[7, 23]. In this case, the change in the variance of log-incomes is -0.25, and the coefficient for the regression of final dollars on initial dollars is $\beta_d = 1.07 > 1$.

To obtain additional possibilities, one can modify how the change in inequality is measured. Consider the following example:

$$[1, 1, 1, 1, 1, 1, 1, 1, 6, 9] \rightarrow [1, 1, 1, 1, 1, 1, 1, 1, 7, 8],$$

The richest person has transferred $1 to the next richest person, which is a clear Lorenz-improvement. Inequality therefore falls by the Lorenz criterion and accordingly for any Lorenz-consistent inequality measure. However, if we measure inequality by the change in the variance of log-incomes, that calculation shows an increase in variance of logs from 0.716 to 0.721 despite the Lorenz-improvement. Moreover, if we regress the change in log-dollars on initial log-dollars, we obtain $\beta_{log} = 1.00045 > 1$, hence divergence in log-dollars. Thus, in this example, log-income changes are divergent, the variance of log-incomes increases (which it must by the Furceri, Wodon-

Yitzhaki theorem), and yet a Lorenz-improvement has taken place. The Lorenz-inconsistency of the variance of log-dollars is on full view here.

The point, then, is that although it is impossible to have both falling inequality as measured by the variance of log-dollars (σ-convergence) and divergent panel income changes as measured by a regression of final log-dollars on initial log-dollars (β-divergence), it *is* possible to have falling inequality *measured some other way* along with divergent income changes *measured some other way*.

3.5 Summary

This section has shown several things:

First, it provided a straightforward multi-person example showing the mutual consistency of rising inequality and convergent panel income changes.

Second, using the formulas for the precise conditions under which a) panel income changes are divergent or convergent in dollars, shares, log-dollars, or proportions, and b) income inequality is rising or falling according to the Lorenz criteria, Lorenz-consistent indices, the variance of log-dollars, and the variance of dollars, it presented a two-person example showing that as the income changes go from most favorable to the initially-richest person to least favorable: a) all divergence/convergence measures decline monotonically and switch from divergent to convergent; b) some divergence/convergence measures switch from divergent to convergent at different places from others; c) the Lorenz pattern is: Lorenz-worsening $\rightarrow$ Lorenz-coincident $\rightarrow$ Lorenz-improvement $\rightarrow$ Lorenz-coincident $\rightarrow$ Lorenz-worsening; d) the three inequality measures are all U-shaped; and e) some inequality measures switch from rising inequality to falling inequality at different places from others.

Third, measuring convergent/divergent income changes in four different ways and looking at whether relative inequality is rising or falling, all thirty-two cells are shown to be possible depending on how relative inequality is measured.

And fourth, a) although it is impossible for divergent income changes and falling relative inequality to both arise when divergence is gauged using log-dollar changes and falling relative inequality is gauged using variance of log-dollars, b) it is possible for divergent income changes and falling relative inequality to coexist when divergence and falling inequality are measured in other ways.

Examples prove possibilities; they do not produce exact conditions. In the next section, we derive a number of necessary and sufficient conditions for each possibility.

4 Mathematical Results

In this section we analytically develop a set of results that establish the connection between changes in relative inequality and our several income change concepts. First, we establish some common notation.

4.1 Notation

Consider an economy with n individuals observed over two time periods, initial (or 0), and final (or 1).

Denote by d_{it} the income of individual i in period t measured in constant monetary units (e.g. real dollars). Whenever possible we will avoid expressing the individual subindex i to avoid clutter. In addition, we will deal with strict monotonic transformations of income, like income as a share of mean income, and log-incomes. We will denote income shares by s, and log-incomes by $\ln d$ respectively.

More generically, when the distinction between these transformations is unimportant we will denote by $y_t = g(d_t)$ the income variable transformed by the strictly monotonically increasing function $g(\cdot)$.[1] Vectors are denoted by bold fonts, e.g. $\mathbf{y_t} = (y_{1t}, y_{2t}, \ldots, y_{nt})$.

Mean income in period t is denoted by μ_t. Income vectors and their transformations are sorted in ascending order of initial-period income. An exception to this is the final income-share vector $\mathbf{s_c}$, where the sorting is ascending in the final-period's income.

LC_t denotes the Lorenz Curve of income in period t, and $LC_1 \succ LC_0$ means that the Lorenz curve in period 1 dominates that of period 0, namely incomes in period 1 are more equally distributed than the ones in period 0 according to the Lorenz-criterion. If the domination is weak we denote it as $LC_1 \succeq LC_0$, which means that incomes in period 1 are at least as equally distributed as those in period 0. Finally, $I(\cdot)$ will be used to denote an arbitrary relative inequality measure.

[1]Since the identity function x=g(x) is a strictly increasing monotonic function too, y can include income in dollars as well.

4.2 Convergence in dollars and rising relative inequality in periods of economic growth

In the previous sections we showed that it is possible to have rising relative inequality and convergent dollar changes when going from period 0 to period 1. In this section we derive the precise conditions for such reconciliation. In everything that follows we consider regressions done in population and abstract from all issues of inference. Consider the final on initial income regression

$$d_1 = \alpha_d + \beta_d d_0 + u_d. \tag{1}$$

The income changes are said to be divergent/convergent in dollars as $\beta_d \gtrless 1$.

We can show the following result:

Proposition 1. ***Relationship Between Divergent/Convergent Dollar Changes, Changing Income Inequality, and Economic Growth.***

Let β_d be defined by the levels regression (1), and denote the correlation coefficient from this regression by r_l. Let $CV(\mathbf{d_0})$ and $CV(\mathbf{d_1})$ denote the initial and final coefficients of variation respectively and let g denote the economy-wide growth rate in incomes between year 0 and year 1. These variables are linked by the following relationship:

$$\beta_d = r_l \frac{CV(\mathbf{d_1})}{CV(\mathbf{d_0})}(1+g). \tag{2}$$

Proof: See Appendix.

Equation 2 gives us a direct way to visualize how rising relative inequality can be reconciled with convergent dollar changes when the economy is growing. In particular, if r_l is small enough or negative then we can have rising relative inequality (i.e. $CV(\mathbf{d_1}) > CV(\mathbf{d_0})$), positive economic growth ($g > 0$) and convergent dollar changes ($\beta_d < 1$) all at the same time. We express this as a corollary.

Corollary 1. ***Imperfect Fit, Rising Inequality, and Convergent Income Changes With Positive Growth.***

In order for rising inequality as measured by the coefficient of variation and convergent income changes in dollars to coexist, the correlation between *final income in dollars* $\mathbf{d_1}$ *and initial income in dollars* $\mathbf{d_0}$ *must be sufficiently low or negative. In particular, rising inequality and convergence in dollars can coexist as long as*

$$r_l < \frac{CV(\mathbf{d_0})}{(1+g)CV(\mathbf{d_1})} < 1.$$

Equation (2) serves to illustrate what we already had seen in our examples, namely that it is possible for divergent income changes ($\beta_d > 1$) and falling relative inequality (i.e. $CV(\mathbf{d_0}) > CV(\mathbf{d_1})$) to coexist when income growth g is strong enough.

Equation (2) also shows that to make rising inequality compatible with convergent dollar changes we must either have economic decline ($g < 0$) or a low r_l. This low level of determination can arise for two reasons: i) a highly non-linear relationship between final and initial incomes, and/or ii) large income changes such that people switch positions as they go from one period to another. In the next sections we show that they key ingredient for this reconciliation is the latter, namely it is crucial to have large income changes that bring positional change.

4.3 Relative inequality changes and divergent/convergent share changes

As mentioned in section 2, the most accepted criterion for judging whether there has been an increase or decrease in relative inequality is the Lorenz Dominance criterion. For this reason we focus most of our attention on this criterion.

Before formulating such criterion we define the ordered final share vector $\mathbf{s_c}$.

Definition. Vector of Final Shares in Ascending Order.
Let $\mathbf{s_c}$ be the counterfactual final income-share vector when incomes are sorted in ascending order of *final* income.

In other words, $\mathbf{s_c}$ has the same distribution of the final vector $\mathbf{s_1}$, but rearranges it in ascending order. Since the final share vector $\mathbf{s_1}$ is ordered according to the *initial* income vector, in general $\mathbf{s_c} \neq \mathbf{s_1}$.

With this notation we can now define the Lorenz Dominance criterion.

Definition. Lorenz Dominance.
Let s_{j0} be the initial income-share of the individual in position j, when shares are sorted in ascending order of initial income. Let s_{jc} be the final income-share of the individual in position j, when shares are sorted in ascending order of final income. The final income distribution Lorenz-dominates the initial one (i.e. $LC_1 \succ LC_0$) whenever

$$\begin{array}{ll} s_{1c} + s_{2c} + \ldots + s_{jc} \geq s_{10} + s_{20} + \ldots + s_{j0} & \text{for j = 1, 2, ..., n - 1 and} \\ s_{1c} + s_{2c} + \ldots + s_{jc} > s_{10} + s_{20} + \ldots + s_{j0} & \text{for some j < n.} \end{array} \tag{3}$$

In other words, having the final period distribution Lorenz-dominate the initial one means that the final distribution is more equally distributed than the initial one according to this criterion. This situation is sometimes also referred as a "Lorenz-improvement" when going from $\mathbf{d_0}$ to $\mathbf{d_1}$. Similarly, if $LC_0 \succ LC_1$ we talk of a "Lorenz-worsening".[2]

Since the Lorenz criterion (3) is formulated in terms of income shares, the natural way to link it with a change regression like the ones used in mobility studies is to compare it to a regression also expressed in shares. In particular, in this section we derive a connection between the Lorenz Dominance criterion (3) and a share-change regression

$$\Delta s = \gamma_s + \delta_s s_0 + u_s. \tag{4}$$

Both conditions (3) and (4) involve initial and final income-shares. However, the final period shares appear sorted differently in the two conditions. In particular, in condition (3), final shares $\mathbf{s_c}$ are sorted in ascending of order of *final* shares, while in condition (4) final shares $\mathbf{s_1}$ are sorted in ascending of order of *initial* shares.

It is easy to show that the sign of the estimated coefficient δ_s in regression (4) is determined by the sign of the covariance[3]

$$cov(\Delta s, s_0) = \frac{\sum_i (s_{1i} - s_{0i}) s_{0i}}{n}.$$

[2] The literature usually expresses condition (3) using income as a share of total income. In order to make an easier link with the regressions involving share changes we express it in terms of shares of mean income. It is obvious that Definition 4.3 is the same in both cases.

[3] Recall average share changes are zero by construction.

Using our counterfactual vector $\mathbf{s_c}$ we can decompose this covariance as

$$cov(\Delta s, s_0) = \frac{\sum_i [(s_{1i} - s_{ci}) + (s_{ci} - s_{0i})] s_{0i}}{n}.$$

That is, whether share changes are convergent or divergent is determined by the sum of two terms:

$$\begin{aligned} W &= \frac{\sum_i [s_{ci} - s_{0i}] s_{0i}}{n} \\ X &= \frac{\sum_i [s_{1i} - s_{ci}] s_{0i}}{n}. \end{aligned} \tag{5}$$

W captures the component of the covariance associated with changes in the shape of the income distribution if positions remain unchanged, and X will capture the component of the covariance associated with positional change, under a fixed marginal distribution. These are sometimes called “structural mobility” and “exchange mobility”, respectively.

We can derive the following two key Lemmas for these terms.

Lemma 1. *Let* $\mathbf{s_0}$ *be the initial vector of shares and* $\mathbf{s_c}$ *be defined as before. Assume that the income transition from period 0 to period 1 involves no crossings of Lorenz Curves. Also, let W be given by equation (5). Then under the stipulated conditions there is Lorenz-worsening if and only if W is positive, i.e.* $LC_1 \prec LC_0 \iff W > 0$.

Proof: See Appendix.

In other words, under the assumption of no Lorenz-crossings, the sign of W fully reflects whether there has been a fall or a rise in inequality judged by the Lorenz-criterion.

Also, in the transition from $\mathbf{s_c}$ to $\mathbf{s_1}$, share changes will be convergent, since in the reranking of individuals there will always be a positive transfer of income shares from a relatively richer individual to a poorer one.

Lemma 2. *Let* $\mathbf{s_c}$ *be defined as before. Let X be given by equation (5). Then* $X \leq 0$.

Proof: See Appendix.

With these two results we can proceed to analyze the connection between share mobility and changes in inequality. For simplicity, let us begin analyzing the case when there is no positional change between initial and final periods.

4.3.1 The case of no change in positions

If all individuals keep their same rank in the initial and final distribution, this is called zero positional change, or synonymously, zero positional mobility. In this case our counterfactual vector $\mathbf{s_c}$ will equal the final share vector $\mathbf{s_1}$, and the sign of δ_s is determined exclusively by W. Given Lemma 1 and the immediate connection between W and δ_s, in the absence of positional changes, the next Proposition follows immediately.

Proposition 2. ***Lorenz Dominance and Share Change Without Positional Change.***

Suppose that when the income distribution vector changes from $\mathbf{d_0}$ to $\mathbf{d_1}$, the change involves

i) no change in positions and

ii) no crossings of Lorenz Curves.

Then Lorenz-worsening occurs if and only if there is divergence in shares, i.e. $LC_1 \prec LC_0 \iff \delta_s > 0$.

Proof: See Appendix.

The intuition (and proof) behind these results is related to a well-known result in the inequality literature stating that a disequalization in the Lorenz sense can be achieved by a series of income transfers from poorer to richer individuals that keep unaltered the individual ranks between the initial and the final periods (see for instance Fields and Fei, 1978). These transfers generate by construction divergent share changes. The exact opposite occurs when there is a Lorenz improvement (i.e. a fall in inequality) and positions remain unchanged.

In other words, as long as we restrict ourselves to the case of no positional mobility and no crossings of Lorenz curves, share mobility and changes in inequality fully align, in the sense that rising inequality only occurs with divergent share-changes and falling inequality only occurs with convergent share-changes.

4.3.2 The case of positional changes

Once we allow for positional changes we need to consider not only the transition from $\mathbf{s_0}$ to $\mathbf{s_c}$, but also from $\mathbf{s_c}$ to $\mathbf{s_1}$. In this last step the shape of the income distribution remains unchanged and pairs of individuals swap positions.

In other words, when going from $\mathbf{s_0}$ to $\mathbf{s_1}$ in the presence of positional changes (but not of Lorenz-crossings) there are two forces at play. The first one, change in the distribution shape with fixed positions, can lead to convergent or divergent share changes, depending on whether there is equalization or disequalization of the anonymous income distribution. The second one, positional rearrangement with a fixed marginal distribution, leads to convergent share-changes always.

In the case of a Lorenz-improvement both components go in the same direction, and share changes are convergent. However, if the income distribution becomes more unequal by the Lorenz-criterion, the two components will move in opposite directions, and depending on which force is dominant there will be convergence or divergence in shares as measured by δ_s in equation (4).

Proposition 3 summarizes the conditions under which each combination of Lorenz-improvement/worsening and convergent/divergent share changes can occur.

Proposition 3. ***Lorenz Dominance and Share Change With Positional Change***
Assume that LC_0 and LC_1 do not cross, and let W and X be defined as in equation (5).

i) If $W > 0, X < 0$, and $W + X < 0$, then there will be share convergence ($\delta_s < 0$) and Lorenz-worsening ($LC_1 \prec LC_0$).

ii) If $W > 0, X \leq 0$, and $W + X > 0$, then there will be share divergence ($\delta_s > 0$) and Lorenz-worsening ($LC_1 \prec LC_0$).

iii) If $W < 0$, and $X \leq 0$, then there will be share convergence ($\delta_s < 0$) and Lorenz-improvement ($LC_1 \succ LC_0$).

iv) It is impossible to simultaneously have a Lorenz-improvement and share divergence.

Proof: See Appendix.

The impossibility in iv) means that for falling inequality to be compatible with divergent share-changes we need to have crossing Lorenz curves. Furthermore, the next corollary follows immediately from Proposition 3.

Corollary 2. *If $\delta_s \geq 0$ then either $LC_0 \succeq LC_1$, or the Lorenz curves of incomes in periods 0 and 1 cross.*

To reiterate in intuitive terms, the results derived in this section: When inequality falls, both the transfers involved for the anonymous equalization and the positional re-ranking are convergent (i.e. from richer to poorer). Yet when inequality rises, there will be disequalizing transfers that change the shape of the income distribution together with equalizing transfers due to the positional swap. Whether there will be convergence or divergence in shares according to equation (4) will depend on which force dominates.

4.4 Mobility in Dollars and Lorenz Dominance

While the previous section establishes a clear connection between change in inequality as gauged by the Lorenz criterion and share changes, on many occasions our interest is not the changes in shares but the changes in dollars. In particular, often when someone is interested in finding out whether "the rich got richer and the poor, poorer" the reference is to changes in dollars and not merely in shares.

In this section we establish a connection relating changes in inequality under Lorenz-dominance and a dollar-change regression

$$\Delta d = \gamma_d + \delta_d d_0 + u_d. \tag{6}$$

In order to derive such a connection, it is useful to express the dollar-change regression (6) in its final-on-initial form (1)

$$d_1 = \alpha_d + \beta_d d_0 + u_d,$$

and recall that in such a case convergence will occur whenever $\beta_d < 1$. Similarly, we can define a final-on-initial share regression

$$s_1 = \alpha_s + \beta_s s_0 + u_s. \tag{7}$$

Using these regressions we can establish the following result.

Lemma 3. *Let μ_t be the mean income in period t, and (β_d, β_s) denote the convergence coefficients given by regressions (1) and (7) in dollars and in shares, respectively. Then*

$$\beta_d = \beta_s \frac{\mu_1}{\mu_0} = \beta_s(1+g).$$

Proof: See Appendix.

Similarly to the previous section, we can derive a series of conditions under which each combination of Lorenz dominance/worsening and convergent/divergent dollar-changes can occur. These conditions are summarized in Table 5 for each growth scenario.[4]

The $(1,1)$ cell in Panel A states that in periods of economic growth, in order to have convergent dollar changes together with Lorenz-worsening, we need to have income changes large enough such that there are positional changes, i.e. $X < 0$ and large enough such that $W + X < 0$. In other words, the low correlation between initial and final period incomes condition in Corollary 1 is due to large income changes that bring positional rearrangements, and not merely due to nonlinearities.

From Table 5 we can also infer a corollary similar to Corollary 2 relating dollar-changes and Lorenz Dominance, as long as there is negative growth (i.e., $g < 0$).

Corollary 3. ***Falling Inequality and Dollar Changes Under Negative Growth.***
In the case $g < 0$, then:

i) Lorenz-improvement implies convergence in dollars, i.e. if $LC_1 \succ LC_0 \Rightarrow \delta_d < 0$.

ii) If $\delta_d \geq 0$ then either $LC_0 \succeq LC_1$, or the Lorenz curves of incomes in periods 0 and 1 cross.

Again, the impossibility result in cell (2,2) in Panel B of Table 5 means that for falling inequality to be compatible with divergent income changes in periods of economic decline we need to have crossing Lorenz curves, as expressed in ii) in Corollary 3.

[4] The conditions can be easily derived from Lemmas 1-3 and Proposition 3. A proof is available from the authors upon request.

What if economic growth is positive? In that case the dollar gains of the initially poor can be smaller than those of the initially rich (which occurs if there is divergence in dollars), yet the share gains of the initially poor can be higher than the share gains of the initially rich. An example is (1, 5) $\rightarrow$ (3, 8).

In more precise terms, in the case of positive economic growth, falling relative inequality will lead to convergence in dollars only if the convergence in shares ($\beta_s < 1$) is stronger than the diverging impact of proportionally rising incomes $(1 + g)$.

4.5 Proportional Income Changes, Dollar Changes and Lorenz-Dominance

4.5.1 Dollar Changes and Proportional Changes

In many applications economists have been interested in studying convergence and divergence in proportional income changes. In particular they have studied whether on average initially rich individual had proportional income changes larger than those of initially poor individuals.

The study of convergent proportional changes is particularly relevant under scenarios of economic decline. To appreciate this, consider a hypothetical two-person economy with the following income transition

$$[2, 50] \rightarrow [1, 45]$$

where the poor individual lost 1 dollar while the rich one lost 5 dollars. By our measure there is convergence in dollars. Yet the 1-dollar loss represented half of the poor individual's income, while the 5-dollar loss represented only a 10% loss for the rich individual. Hence, in this example there was convergence in dollars but divergence in proportional changes.

Convergence in dollars and divergence in proportional changes cannot coexist in periods of economic growth, since if the initially poor gain more in dollars than the initially rich (i.e., there is convergence in dollars) then proportional changes are necessarily convergent as well. In this section we explore the relationship between proportional changes in income and changes in inequality that respect Lorenz-Dominance.

4.5.2 Log-Income Approximation

The most common way to measure proportional convergence is by approximating proportional changes by changes in log-income and estimating a double-log regression

$$\Delta \ln d = \gamma_{log} + \delta_{log} \ln d_0 + u_{log} \tag{8}$$

or its equivalent "final on initial" form $\ln d_1 = \alpha_{log} + \beta_{log} \ln d_0 + u_{log}$.

As shown in section 3, we can find all possible combinations of falling/rising inequality with convergent/divergent log-income changes. In particular, contrary to the share-change case we can find examples that make compatible falling inequality as gauged by a Lorenz-improvement and divergent log-income changes.

The reason why there is no contradiction between divergence in log-incomes and falling inequality (as caused by a progressive transfer) is closely related to the reasons why the variance of log-incomes is not a Lorenz-consistent measure of inequality. In particular, log-incomes can be divergent even if proportional changes are convergent if a progressive transfer occurs sufficiently high up in the income distribution. To derive the precise conditions under which this happens, we need first the following definition.

Definition. Equalizing Rank-Preserving Transfer.
A rank-preserving equalizing transfer $h > 0$ is a transfer of income shares between two individuals with ranks i and j for $i > j$, such that:

$$\begin{aligned}
&d_{0k} = d_{1k} && \text{for } k \neq i, j,\\
&d_{1i} = d_{0i} - h, &&\\
&d_{1j} = d_{0j} + h, && \text{where:}\\
&\text{if } i = j+1, && h < (d_{0i} - d_{0j})/2;\\
&\text{if } i > j+1, && h < \min[(d_{0j+1} - d_{0j}), (d_{0i} - d_{0i-1})].
\end{aligned}$$

A rank-preserving disequalizing transfer is defined similarly. We can show the following lemma for a single transfer that is sufficiently small that it preserves ranks:

Lemma 4. ***A Single Rank-Preserving Transfer and Log-income Changes.***
Let gm denote the geometric mean of income at period 0, and $\exp(1) = 2.718$.

Consider two individuals i and j such that $d_{0i} > d_{0j} > gm * \exp(1)$. Let $h > 0$ be a sufficiently small rank-preserving transfer between i and j. Under these conditions,

a) if h is equalizing, then $LC_1 \succ LC_0$ and $\delta_{log} > 0$.

b) if h is disequalizing, then $LC_1 \prec LC_0$ and $\delta_{log} < 0$.

Proof: See Appendix.

In other words, it would be easy to misinterpret a log-change regression like (8). The log-change regression can indicate divergence, as we define it, even when the income changes lead to a Lorenz-improvement. Rank-preserving equalizations which occur sufficiently high up in the income distribution can lead to divergence in log-dollars. Cowell (2011) shows that the variance of log-incomes will mis-rank which distribution is more unequal vis-à-vis the Lorenz ranking under a similar condition.[5]

4.5.3 True Proportional Changes

One alternative to the log-income changes regression (8) is to study proportional changes by regressing the true proportional change in incomes on initial income, namely

$$\text{pch} \equiv (d_1 - d_0)/d_0 = \phi + \theta d_0 + e. \tag{9}$$

In this case, if θ is positive we will say there is divergence in proportions, if θ is negative we will say there is convergence in proportions, and if θ equals zero we will say the proportional changes are equal with respect to initial income.

In the case of regression (9) we can show the following results linking inequality changes and true proportional changes.

Proposition 4. ***Falling Inequality and True Proportional Changes with Possible Positional Changes.***

If there is a Lorenz-improvement when going from period 0 to period 1, then there must be convergence in true proportional changes, i.e. if $LC_1 \succ LC_0$, then $\theta < 0$.

[5] It can also be shown that if incomes follow a log-normal or a Pareto distribution then such misinterpretations by (8) cannot arise. A proof is available from the authors upon request.

Proof: See Appendix.

Corollary 4. ***Implication of Divergent Proportional Changes.***
If the true proportional changes are non-convergent ($\theta \geq 0$) *then either*

i) a weak Lorenz-worsening has taken place $LC_0 \succeq LC_1$*, or*

ii) the Lorenz curves of incomes in periods 0 and 1 cross.

Furthermore, we can establish a precise condition for when there will be a Lorenz-worsening despite the existence of convergent proportional changes.

Corollary 5. ***Lorenz-Worsening and Convergent Proportional Changes.*** *Let* $\mathbf{s_c}$ *be defined as before. Assume there is Lorenz-worsening* ($LC_0 \succeq LC_1$)*. If*

$$\frac{1}{n}\sum_i \frac{s_{1i} - s_{ci}}{s_{0i}} > \left|\frac{1}{n}\sum_i \frac{s_{ci} - s_{0i}}{s_{0i}}\right|,$$

then proportional changes will be convergent.

Proof: See Appendix.

The intuition is the same as before: if income changes are large enough, and in a suitable pattern, we can have rising inequality, positional changes, and convergent proportional changes.

4.6 Convergence and Changes in the Variance

4.6.1 Convergence of an income variable y and the change in the variance of y

In the macro and labor literatures, it is quite common to assess changes in relative inequality by focusing on the variance of log-incomes. In spite of its Lorenz-inconsistency (as already noted in Section 2), the variance of logs remains quite popular in the literature. In this section we present the basic relationship between changes in inequality as measured by the variance of logs and the coefficient in a log-change regression (8). This result was derived independently by Furceri (2005) and Wodon and Yitzhaki (2006), and we present it next.

Lemma 5. ***Log-Income Convergence and Variance of Log-Income.*** *If* $\Delta V(\ln d) < 0$*, then* $\delta_{log} < 0$*, for* δ_{log} *defined by a regression (8).*

Rather than presenting a proof of this result, we include a more general result linking the variance of any monotonically increasing function of income $y = g(d)$ (e.g. logarithms, shares, dollars, etc.) and the coefficient of a regression of the changes in this generic variable y and the initial level it takes y_0. Namely, we present a result concerning the relationship between the changes in $V(g(d))$ and the coefficient δ_y in a regression of Δy on y_0

$$\Delta y = \gamma_y + \delta_y y_0 + u_y \tag{10}$$

for the case of any monotonically increasing function $g(d)$ of income:

Proposition 5. ***Linear Convergence and Changes in Variance.***
If $\Delta V(y) < 0$, then $\delta_y < 0$.

Proof: See Appendix.

As before the contrapositive of this proposition is also a useful way of visualizing this result.

Corollary 6. ***Implication of Divergent Changes on Changes in Variance.***
If $\delta_y \geq 0$ then $\Delta V(y) \geq 0$.

Proposition 5 and Corollary 6 show that divergence in the changes of a monotonically increasing function of income $y = g(d)$ implies a rising variance of this function, or alternatively a falling variance of $y = g(d)$ implies convergent changes of $y = g(d)$. However, convergence does not imply a falling variance: $\delta_y < 0 \nRightarrow \Delta V(y) < 0$.

We can also establish a precise condition for when we can observe rising inequality, as measured by the variance of y, and convergent changes.

Corollary 7. ***Convergent Changes and Rising Variances.***

Let β_y be the coefficient in a final-on-initial regression $y_1 = \alpha_y + \beta_y y_0 + u_y$ for a generic income variable y. If $\beta_y < 1$ and

$$1 - \beta_y^2 < \frac{V(u_y)}{V(y_0)},$$

then $\Delta V(y) \geq 0$.

Proof: See Appendix.

To reemphasize these results pertain to any monotonically increasing function of income, as long as we use the <u>same</u> function of income $y = g(d)$ as dependent and independent variables, i.e. as long as we run share-changes on initial shares, log-income changes on initial log-incomes, etc.

4.7 Extensions to Cases of a Single-Lorenz Crossing from above

So far, many of our results have been derived by analyzing rising or falling inequality as judged by Lorenz-worsenings or improvements. In practice, it is common to find that the Lorenz curves of two income distributions cross.

The extension of our results to cases of Lorenz-crossings is not straightforward, as some of the previous counter-examples have shown. However, we can still find a relationship between convergence coefficients δ and certain inequality measures under a particular type of Lorenz-crossing. In particular, we will focus on a single crossing from above as defined next.

Definition. Single Lorenz Crossing From Above.

Denote by $LC(\mathbf{d}; p)$ the Lorenz curve ordinate corresponding to the lowest 100p% of income recipients for $p \in [0, 1]$. The Lorenz curve for a distribution $\mathbf{d}$ is said to intersect that of $\mathbf{d}'$ *once from above* iff there exists $p* \in (0, 1)$ and intervals $P \equiv [0, p*]$ and $P' \equiv [p*, 1]$ such that

$$LC(\mathbf{d}; p) \geq LC(\mathbf{d}'; p) \quad \forall p \in P \quad \text{and} \; > \; \text{for some } p \in P$$
$$LC(\mathbf{d}; p) \leq LC(\mathbf{d}'; p) \quad \forall p \in P' \quad \text{and} \; < \; \text{for some } p \in P'.$$

To better understand the welfare properties of the inequality assessments under this type of crossing we need to define the property of "transfer sensitivity" of an inequality measure.

Definition. Transfer-Sensitive Inequality Measures (Shorrocks and Foster, 1987)
An inequality measure $I()$ is transfer sensitive **(TS)** iff $I(\mathbf{d_0}) > I(\mathbf{d_1})$ whenever $\mathbf{d_1}$ is obtained from $\mathbf{d_0}$ by a series of transfers whereby at each stage i) a progressive transfer occurs at lower income levels, ii) a regressive transfer occurs at higher income levels, iii) ranks remain unchanged, and iv) the variance of incomes remains unchanged.

Intuitively speaking, a transfer-sensitive inequality measure is one that records a fall in inequality whenever there is a progressive transfer at the lower part of the income distribution in tandem with a regressive transfer at higher income levels, to the extent that the transfers are comparable in the sense required by condition iv) in the above definition.[6]

[6] A formal statement together with a careful discussion of the concept is presented in Shorrocks and Foster (1987).

In other words, transfer-sensitive inequality measures can rank certain types of distributions in the presence of Lorenz-crossings by giving greater weight to transfers that occur in the lower part of the income distribution. Shorrocks and Foster (1987) show that the Atkinson family and certain members of the Generalized Entropy class satisfy this property, but the Gini coefficient does not.

An important result due to Shorrocks and Foster (1987) provides a relation between transfer-sensitive inequality indices and the Coefficient of Variation, CV. For the sake of completeness we reproduce it next without proof.

Proposition 6. ***Transfer Sensitive Inequality Indices and the Coefficient of Variation.*** *(Shorrocks and Foster, 1987)*
If the Lorenz curve of $\mathbf{d_1}$ *intersects that of* $\mathbf{d_0}$ *once from above, then* $I(\mathbf{d_0}) > I(\mathbf{d_1})$ *for all inequality measures* $I_{TS}()$ *satisfying transfer sensitivity* ***(TS)****, scale-independence* ***(S)****, and population-independence* ***(P)*** $\iff CV(\mathbf{d_0}) \geq CV(\mathbf{d_1})$.

Since the CV^2 is the variance of shares,[7] we can use Propositions 5 and 6 to establish a result linking the coefficient in a share regression, i.e. δ_s, to the changes in Transfer Sensitive inequality indices, I_{TS}, when there is a single-crossing from above in Lorenz curves.

Proposition 7. ***Single Lorenz-Crossing from Above and Share Changes***
Let $I_{TS}()$ *denote all inequality measures satisfying transfer sensitivity* ***(TS)****, scale-independence* ***(S)****, and population-independence* ***(P)****. If*

i) the Lorenz curve of $\mathbf{d_1}$ *intersects that of* $\mathbf{d_0}$ *once from above and*

ii) $CV(\mathbf{d_0}) \geq CV(\mathbf{d_1})$,

then both $\delta_s \leq 0$ *and* $I_{TS}(\mathbf{d_0}) > I_{TS}(\mathbf{d_1})$.

Proof: See Appendix.

This result is weaker than Proposition 3.*iv*), because it does not say that a single crossing from above will imply share-convergence. Instead it states that if the Lorenz curve of final-period incomes crosses that of initial incomes once from above *and* the Coefficient of Variation falls, then there will be both share-convergence and a fall in inequality according to all relative inequality

[7] This is shown in the proof of Proposition 7.

measures that satisfy transfer sensitivity **(TS)**, scale independence **(S)**, and population independence **(P)**.

To illustrate empirically Proposition 7 consider the transition

$$\mathbf{d_0} = [1, 5, 10, 11] \rightarrow \mathbf{d_1} = [2, 4, 9, 12].$$

In this case the conditions of the Proposition are satisfied, namely there is: i) a single-crossing from above in the Lorenz curves, and ii) a falling CV. In this case it is readily verified that commonly used indices like the Atkinson family and Generalized Entropy with parameter < 2 will mark a reduction in inequality, and there is share convergence as well.

To appreciate the importance of having a falling Coefficient of Variation (condition ii) in Proposition 7), consider the example $\mathbf{d_0} = [1, 5, 10] \rightarrow \mathbf{d_1} = [2, 4, 25]$. In this case the Lorenz curve of $\mathbf{d_1}$ crosses that of $\mathbf{d_0}$ once from above, yet the Coefficient of Variation does not fall, and we cannot appeal to Proposition 7.

5 Conclusion

This paper has explored the relationship between changing income inequality in the cross section and panel income changes in times of economic growth and economic decline. We began by presenting examples showing that nearly all combinations of rising or falling income inequality, convergent or divergent panel income changes, and economic growth or decline are possible (Table 4). Then, using precise conditions for each of the above (Table 2), we derived a host of mathematical results, which are summarized for convenience in Table 6. Each cell of the table contains to the proposition, corollary, or lemma where the result is derived.

Three observations are particularly trenchant.

First, the great majority of results are derived measuring inequality change by Lorenz-improvements and Lorenz-worsenings. Thus, all who agree on the desirability of using Lorenz criteria for making inequality comparisons would feel confident that the various combinations involve "good" ways of measuring inequality.

Second, some of the results require Lorenz crossings and hold for a carefully chosen inequality index but not for all Lorenz-consistent indices. Consequently, these results are weaker than those based on Lorenz-dominance.

And third, some combinations are impossible, but there are very few of them. One impossibility is the one previously proved by Furceri (2005) and Wodon and Yitzhaki (2006) and confirmed here: that it is impossible to have divergent log-dollar changes and falling relative inequality as measured by the variance of logs. However, it is possible to have divergent log-dollar changes and a Lorenz-improvement, hence falling inequality as measured by *any Lorenz-consistent index*. The prior impossibility result is due to the authors' use of an inequality index which is *not Lorenz-consistent*. Two other impossibilities are ones which we have proven here and were not previously in the literature (Proposition 3). One is that we cannot simultaneously have divergent panel income changes in shares and a Lorenz-improvement, either in times of economic growth or in times of economic decline. The other is that in times of economic decline we cannot simultaneously have divergent panel income changes in dollars and a Lorenz-improvement.

Every other combination of rising or falling income inequality, divergent or convergent panel income changes, and economic growth or decline is possible, and we have displayed the conditions under which each arises.

Finally, let us return to where we started: How can rising income inequality and convergent panel income changes both take place in times of economic growth? The answer is given by Propositions 1-Corollary 3 and by Table 5. In particular, these results highlight that for these two seemingly contradictory facts to be reconciled, one needs large panel income changes such that some initially low-earners will become high earners in a widening distribution. A consequence of this is that if we divide the population in groups according to initial income, the within-group variance of income changes will be large.

Applied to panel data on people, these conditions often apply, which is why rising income inequality and convergent panel changes in dollars have been found in many countries. However, applied to panel data on countries, it appears that these conditions do not apply, which is why we do not see the poorer countries getting ahead in dollars at a faster rate than richer countries.

The results derived in this paper open up a series of questions as to the nature of these individual income changes. For instance, when rising inequality is observed together with convergent panel income changes, is this finding driven by a few individuals experiencing large changes, or by many individuals experiencing moderate changes, or are both important? Exploring the precise way in which these large individual changes occur is an important question for future research.

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Tables

Table 1: Possibilities for Rising/Falling Inequality and Convergent/Divergent Panel Income Changes

	Rising Inequality	Falling Inequality
Convergent Panel Income Changes	✓	✓
Divergent Panel Income Changes	✓	✓

✓: This cell is possible both in times of economic growth and in times of economic decline.

Table 2: Conditions for Rising/Falling Relative Income Inequality And Divergent/Convergent Panel Income Changes

Part A: Conditions for Rising/Falling Income Inequality

Lorenz Criteria Definition of Lorenz dominance:
Let s_{jt} be the income share of the individual in position j of the income distribution of period t, when such distribution is sorted in ascending order of period 0 income. Let s_{jc} be the income share of the individual in position j of the income distribution of the final period 1, when such distribution is sorted in ascending order of period's 1 income. We say that the final income distribution Lorenz-dominates the initial one (i.e., $LC_1 \succ LC_0$) whenever

$$s_{1c} + s_{2c} + \ldots + s_{jc} \geq s_{10} + s_{20} + \ldots + s_{j0} \quad \text{for j = 1, 2, ..., n - 1 and}$$
$$s_{1c} + s_{2c} + \ldots + s_{jc} > s_{10} + s_{20} + \ldots + s_{j0} \quad \text{for some j < n.}$$

Definitions of Lorenz-improvement, Lorenz-worsening, and Lorenz-crossing:
Lorenz-improvement over time: $LC_1 \succ LC_0$ - relative inequality falls by the Lorenz criterion.
Lorenz-worsening over time: $LC_0 \succ LC_1$ - relative inequality rises by the Lorenz criterion.
Lorenz-crossing: Neither $LC_1 \succ LC_0$ nor $LC_0 \succ LC_1$.
Change in Lorenz-Consistent Indices
An inequality index is Lorenz-consistent if it weakly increases (decreases) whenever there is a Lorenz-worsening (Lorenz-improvement).
In a case of Lorenz crossing, at least one Lorenz-consistent index registers rising inequality and at least one other one registers falling inequality.
Change in Variance of Log-Dollars
The variance of log-dollars, though relative, is not Lorenz-consistent.
Change in Variance of Dollars
The variance of dollars is not relative, therefore is not Lorenz-consistent.

Part B: Conditions for Divergent/Convergent Income Changes

A generic final on initial regression: $y_1 = \alpha_y + \beta_y y_0 + u_y$. Call this "final on initial regression".
A generic changes on initial regression: $\Delta y = \gamma_y + \delta_y y_0 + u_y$. Call this "change regression".
Proportional changes regression: pch d $\equiv (d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}$.

Dollars
Denote income in dollars in period t by d_t for t=0,1.
Replace y_t by d_t in the above regressions, then:
"Divergent in dollars" if $\beta_d > 1$ (or $\delta_d > 0$).
"Convergent in dollars" if $\beta_d < 1$ (or $\delta_d < 0$).

Shares
Denote income share in period t by $s_t \equiv d_t/\mu_t$ for t=0,1.
Replace y_t by s_t in the above regressions, then:
"Divergent in shares" if $\beta_s > 1$ (or $\delta_s > 0$).
"Convergent in shares" if $\beta_s < 1$ (or $\delta_s < 0$).

Log-Dollars
Denote income in log-dollars in period t by $\ln d_t$ for t=0,1.
Replace y_t by $\ln d_t$ in the above regressions, then:
"Divergent in log-dollars" if $\beta_{log} > 1$ (or $\delta_{log} > 0$).
"Convergent in log-dollars" if $\beta_{log} < 1$ (or $\delta_{log} < 0$).

Proportional Changes
Proportional change in dollars on initial dollars regression:
Define proportional change in income as pch d $\equiv (d_1 - d_0)/d_0$.
Regress proportional change in income on initial income to obtain
pch d $= \phi + \theta d_0 + u_{pch}$, then:
"Divergent in proportional changes" if $\theta > 0$.
"Convergent in proportional changes" if $\theta < 0$.

Table 3: Rising / Falling Inequality and Divergent / Convergent Income Changes in a Two person Example

Initial Income		Income Change		Final Income		A. Rising/Falling Income Inequality				B. Regression Coefficients Divergent/Convergent Panel Income Changes			
Person		Person		Person		Lorenz	Change in CV^2	Change in Variance of Log-Dollars	Change in Variance of Dollars	Dollars (D)	Shares (S)	Log Dollars (L)	Proportional (P)
1	2	1	2	1	2								
5	20	-5	10	0	30	**Lorenz-worsening**	**1.28**		**337.5**	2.00	1.67		0.10
5	20	-4	9	1	29	**Lorenz-worsening**	**1.02**	**4.71**	**279.5**	1.87	1.56	2.43	0.08
5	20	-3	8	2	28	**Lorenz-worsening**	**0.78**	**2.52**	**225.5**	1.73	1.44	1.90	0.07
5	20	-2	7	3	27	**Lorenz-worsening**	**0.56**	**1.45**	**175.5**	1.60	1.33	1.58	0.05
5	20	-1	6	4	26	**Lorenz-worsening**	**0.36**	**0.79**	**129.5**	1.47	1.22	1.35	0.03
5	20	0	5	5	25	**Lorenz-worsening**	**0.17**	**0.33**	**87.5**	1.33	1.11	1.16	0.02
5	20	1	4	6	24	Lorenz-coincidence	0.00	0.00	**49.5**	1.20	1.00	1.00	0.00
5	20	2	3	7	23	*Lorenz-improvement*	*-0.15*	*-0.25*	**15.5**	1.07	***0.89***	***0.86***	***-0.02***
5	20	3	2	8	22	*Lorenz-improvement*	*-0.28*	*-0.45*	*-14.5*	***0.93***	***0.78***	***0.73***	***-0.03***
5	20	4	1	9	21	*Lorenz-improvement*	*-0.40*	*-0.60*	*-40.5*	***0.80***	***0.67***	***0.61***	***-0.05***
5	20	5	0	10	20	*Lorenz-improvement*	*-0.50*	*-0.72*	*-62.5*	***0.67***	***0.56***	***0.50***	***-0.07***
5	20	6	-1	11	19	*Lorenz-improvement*	*-0.58*	*-0.81*	*-80.5*	***0.53***	***0.44***	***0.39***	***-0.08***
5	20	7	-2	12	18	*Lorenz-improvement*	*-0.64*	*-0.88*	*-94.5*	***0.40***	***0.33***	***0.29***	***-0.10***
5	20	8	-3	13	17	*Lorenz-improvement*	*-0.68*	*-0.92*	*-104.5*	***0.27***	***0.22***	***0.19***	***-0.12***
5	20	9	-4	14	16	*Lorenz-improvement*	*-0.71*	*-0.95*	*-110.5*	***0.13***	***0.11***	***0.10***	***-0.13***
5	20	10	-5	15	15	*Lorenz-improvement*	*-0.72*	*-0.96*	*-112.5*	***0.00***	***0.00***	***0.00***	***-0.15***
5	20	11	-6	16	14	*Lorenz-improvement*	*-0.71*	*-0.95*	*-110.5*	***-0.13***	***-0.11***	***-0.10***	***-0.17***
5	20	12	-7	17	13	*Lorenz-improvement*	*-0.68*	*-0.92*	*-104.5*	***-0.27***	***-0.22***	***-0.19***	***-0.18***
5	20	13	-8	18	12	*Lorenz-improvement*	*-0.64*	*-0.88*	*-94.5*	***-0.40***	***-0.33***	***-0.29***	***-0.20***
5	20	14	-9	19	11	*Lorenz-improvement*	*-0.58*	*-0.81*	*-80.5*	***-0.53***	***-0.44***	***-0.39***	***-0.22***
5	20	15	-10	20	10	*Lorenz-improvement*	*-0.50*	*-0.72*	*-62.5*	***-0.67***	***-0.56***	***-0.50***	***-0.23***
5	20	16	-11	21	9	*Lorenz-improvement*	*-0.40*	*-0.60*	*-40.5*	***-0.80***	***-0.67***	***-0.61***	***-0.25***
5	20	17	-12	22	8	*Lorenz-improvement*	*-0.28*	*-0.45*	*-14.5*	***-0.93***	***-0.78***	***-0.73***	***-0.27***
5	20	18	-13	23	7	*Lorenz-improvement*	*-0.15*	*-0.25*	**15.5**	***-1.07***	***-0.89***	***-0.86***	***-0.28***
5	20	19	-14	24	6	Lorenz-coincidence	0.00	0.00	**49.5**	***-1.20***	***-1.00***	***-1.00***	***-0.30***
5	20	20	-15	25	5	**Lorenz-worsening**	**0.17**	**0.33**	**87.5**	***-1.33***	***-1.11***	***-1.16***	***-0.32***
5	20	21	-16	26	4	**Lorenz-worsening**	**0.36**	**0.79**	**129.5**	***-1.47***	***-1.22***	***-1.35***	***-0.33***
5	20	22	-17	27	3	**Lorenz-worsening**	**0.56**	**1.45**	**175.5**	***-1.60***	***-1.33***	***-1.58***	***-0.35***
5	20	23	-18	28	2	**Lorenz-worsening**	**0.78**	**2.52**	**225.5**	***-1.73***	***-1.44***	***-1.90***	***-0.37***
5	20	24	-19	29	1	**Lorenz-worsening**	**1.02**	**4.71**	**279.5**	***-1.87***	***-1.56***	***-2.43***	***-0.38***
5	20	25	-20	30	0	**Lorenz-worsening**	**1.28**		**337.5**	***-2.00***	***-1.67***		***-0.40***

Bold text indicates rising inequality. *Italicized text* indicates falling inequality. ***Bold italicized text*** indicates convergent panel income changes.
`Teletype text` indicates divergent panel income changes.
(D) Regression of Final Dollars on Initial Dollars. (S) Regression of Final Share on Initial Share. (L) Regression of Final Log-Dollars on Initial Log-Dollars.
(P) Regression of Proportional Changes on Initial Dollars.

Table 4: Examples of Possibilities in Times of Economic Growth and Decline.

Final on Initial Regression: $y_1 = \alpha_y + \beta_y y_0 + u_y$
Changes Regression: $\Delta y_1 = \gamma_y + \delta_y y_0 + u_y$
Proportional Changes Regression: $\frac{d_1 - d_0}{d_0} = \phi + \theta d_0 + u_{pch}$

		Economic Growth Positive		Economic Growth Negative	
		Rising Relative Inequality	**Falling Relative Inequality**	**Rising Relative Inequality**	**Falling Relative Inequality**
Convergence/divergence according to linear regression coefficient	**Convergent**				
	Dollar changes ($\beta_d < 1 \iff \delta_d < 0$)	$[5,20]\rightarrow[25,5]^{LW}$	$[5,20]\rightarrow[10,20]^{LI}$	$[7,23]\rightarrow[20,5]^{LW}$	$[5,25]\rightarrow[5,20]^{LI}$
	Share changes ($\beta_s < 1 \iff \delta_s < 0$)	$[5,20]\rightarrow[25,5]^{LW}$	$[5,20]\rightarrow[10,20]^{LI}$	$[7,23]\rightarrow[20,5]^{LW}$	$[5,25]\rightarrow[5,20]^{LI}$
	Log-dollar changes ($\beta_{log} < 1 \iff \delta_{log} < 0$)	$[5,20]\rightarrow[25,5]^{LW}$	$[5,20]\rightarrow[10,20]^{LI}$	$[7,23]\rightarrow[20,5]^{LW}$	$[5,25]\rightarrow[5,20]^{LI}$
	Proportional changes ($\theta < 0$)	$[5,20]\rightarrow[25,5]^{LW}$	$[5,20]\rightarrow[10,20]^{LI}$	$[7,23]\rightarrow[20,5]^{LW}$	$[5,25]\rightarrow[5,20]^{LI}$
	Divergent				
	Dollar changes ($\beta_d > 1 \iff \delta_d > 0$)	$[5,20]\rightarrow[5,25]^{LW}$	$[5,20]\rightarrow[7,23]^{LI}$	$[10,20]\rightarrow[5,20]^{LW}$	$[2,9,18]\rightarrow [2,6.1,18]^{LX^*}$
	Share changes ($\beta_s > 1 \iff \delta_s > 0$)	$[5,20]\rightarrow[5,25]^{LW}$	$[1,5,10]\rightarrow [2,4,25]^{LX^*}$	$[10,20]\rightarrow[5,20]^{LW}$	$[0.6,3.2,10]\rightarrow [0.54,1.5,8.76]^{LX^*}$
	Log-dollar changes ($\beta_{log} > 1 \iff \delta_{log} > 0$)	$[5,20]\rightarrow[5,25]^{LW}$	$[1,1,1,1,1,1,1,1,6,9]\rightarrow [1,1,1,1,1,1,1,1,7,8]^{LI^x}$	$[10,20]\rightarrow[5,20]^{LW}$	$[2,3.1,9]\rightarrow [2,2.2,8.5]^{LX^*}$
	Proportional changes ($\theta > 0$)	$[5,20]\rightarrow[5,25]^{LW}$	$[1,5,10]\rightarrow [2,4,25]^{LX^*}$	$[10,20]\rightarrow[5,20]^{LW}$	$[0.6,3.2,10]\rightarrow [0.54,1.5,8.76]^{LX^*}$

Notes: LW: Lorenz worsening, LI: Lorenz improvement, LX: Lorenz crossing
*: If measure changing inequality by income share of the poorest tercile.
x: This cell is impossible if inequality is measured by change in logs.

Table 5: Conditions for Convergent/Divergent Panel Income Changes and Lorenz Dominance

Panel A: Economic Growth ($g > 0$)

	Lorenz-Worsening	Lorenz-Improvement
Convergent Panel Income Changes	$W > 0,\ X < 0,\ W + X < 0$ $\beta_s < 1/(1+g) < 1$	$X \leq 0,\ W < 0$ $\beta_s < 1/(1+g) < 1$
Divergent Panel Income Changes	$W > 0$ $1/(1+g) < \min\{1, \beta_s\}$	$X \leq 0,\ W < 0$ $1/(1+g) < \beta_s < 1$

Panel B: Economic Decline ($g < 0$)

	Lorenz-Worsening	Lorenz-Improvement
Convergent Panel Income Changes	$W > 0$ $\max\{1, \beta_s\} < 1/(1+g)$	$X \leq 0,\ W < 0$ $\beta_s < 1 < 1/(1+g)$
Divergent Panel Income Changes	$W > 0, X \leq 0, (W + X) > 0$ $\beta_s > 1/(1+g) > 1$	Impossible

Table 6: Summary of Results

		Economic Growth Positive		Economic Growth Negative	
		Rising Relative Inequality	**Falling Relative Inequality**	**Rising Relative Inequality**	**Falling Relative Inequality**
Convergence/divergence according to linear regression coefficient	**Convergent**				
	Dollar changes	Prop. 1	Prop. 1	Prop. 1	Prop. 1
		Table 5 panel A (1,1)	Table 5 panel A (1,2)	Table 5 panel B (1,1)	Table 5 panel B (1,2)
	Share changes	Prop. 3.i)	Prop. 2 & 3.iii)	Prop. 3.i)	Prop. 2 & 3.iii)
	Log-dollar changes	Lemma 4	Lemma 5	Lemma 4	Lemma 5
		Cor. 7		Cor. 7	
	Proportional changes	Cor. 5	Prop. 4	Cor. 5	Prop. 4
	Divergent				
	Dollar changes	Prop. 1	Prop. 1	Prop. 1	Prop. 1
		Table 5 panel A (2,1)	Table 5 panel A (2,2)	Table 5 panel B (2,1)	Lorenz-crossings needed
	Share changes	Prop. 2 & Cor. 2	Cor. 2	Prop. 2 & Cor. 2	Cor. 2
		Prop. 3.ii)	Lorenz-crossings needed	Prop. 3.ii)	Lorenz-crossings needed
	Log-dollar changes	Cor. 6	Lemma 4 & 5	Cor. 6	Lemma 4 & 5
	Proportional changes	Cor. 4	Cor. 4	Cor. 4	Cor. 4
			Lorenz-crossings needed		Lorenz-crossings needed

Notes: Prop. refers to Proposition
Cor. refers to Corollary
Table # (i,j) refers to cell (i,j) of the panel within the Table.

Appendix

Proofs of Propositions

Proposition 1. By definition

$$r_l = \frac{cov(d_1, d_0)}{\sqrt{V(d_1)}\sqrt{V(d_0)}}$$

and

$$\beta_d = r_l \frac{\sqrt{V(d_1)}}{\sqrt{V(d_0)}}.$$

However,

$$\begin{aligned}\frac{\sqrt{V(d_1)}}{\sqrt{V(d_0)}} &= \frac{\sqrt{V(d_1)}/\mu_1}{\sqrt{V(d_0)}/\mu_0}\frac{\mu_1}{\mu_0} \\ &= \frac{CV(\mathbf{d_1})}{CV(\mathbf{d_0})}\frac{\mu_1}{\mu_0}.\end{aligned}$$

Moreover,

$$\mu_1 = (1+g)\mu_0$$

where g is the economy-wide income growth rate. Combining these equations together we obtain equation (2). □

Lemma 1. a) Lorenz Worsening implies $W > 0$.

First, rank individuals in ascending order according to their initial level of income.

Define a <u>rank-preserving disequalizing transfer</u> $h > 0$ as a transfer of income shares between two individuals with ranks i and j for $i < j$, such that:

$$\begin{aligned} s_{0k} &= s_{ck} && \text{for } k \neq i, j, \\ s_{ci} &= s_{0i} - h, && \\ s_{cj} &= s_{0j} + h, && \text{where:} \\ \text{if } j &\geq i+1, && h \leq min[(s_{0j+1} - s_{0j}), (s_{0i} - s_{0i-1})]. \end{aligned}$$

Theorem 2.1 in Fields and Fei (1978) implies that if the distribution of $\mathbf{s_0}$ Lorenz-dominates that of $\mathbf{s_c}$, i.e. if $LC_0 \succ LC_c$, then it is possible to go from $\mathbf{s_0}$ to $\mathbf{s_c}$ by means of a series of rank-preserving disequalizing transfers, like the one above defined.

One convenient way of representing such transfers is by indexing them as h_{ij} where the first index, i, indicates who is making a transfer and the second one, j, who is receiving it.

Since the transfers are disequalizing, and no one makes a transfer to himself, they satisfy the conditions:

$$\begin{aligned} &h_{ij} = 0 \quad \text{for } i \geq j \\ &h_{ij} \geq 0 \quad \text{for } i < j \quad \text{with strict inequality for some individuals.} \end{aligned}$$

The total transfers made by individual i will be the sum over the second index j, namely

$$h_{i\cdot} = \sum_{j=1}^{n} h_{ij}.$$

Similarly, the total transfers received by this same individual will be the sum over the first index, namely

$$h_{\cdot i} = \sum_{j=1}^{n} h_{ji}.$$

Hence, the change in this person's income share can be expressed as the difference in the two previous quantities, i.e.

$$s_{ci} - s_{0i} = h_{\cdot i} - h_{i\cdot} = \sum_{j=1}^{n} h_{ji} - \sum_{j=1}^{n} h_{ij}.$$

By construction, the sum of the share changes over all individuals is zero, hence each person's share loss is somebody else's share gain, and also each share gain is somebody else's loss. In other words, the transfers h_{ij} appear with a positive sign in the share change of individual j, and with a negative sign in the share change of individual i. Furthermore, the sender i is always poorer than the receiver j, since the transfer is disequalizing. Hence, for each transfer h_{ij} we have

$$h_{ij}s_{0j} - h_{ij}s_{0i} = h_{ij}(s_{0j} - s_{0i}) \geq 0.$$

Hence, W can be rewritten as

$$\begin{aligned} W &= n^{-1}\sum_i (s_{ci} - s_{0i})s_{0i} \\ &= n^{-1}\sum_i \left(\sum_{j=1}^n h_{ji} - \sum_{j=1}^n h_{ij} \right) s_{0i}. \end{aligned}$$

That is, W will be the average of terms $h_{ij}(s_{0j} - s_{0i})$ for all the transfers h_{ij}. Since all these terms are non-negative, and some will be strictly positive, then W will be positive.

In other words, we have shown that $LC_0 \succ LC_c$ implies $W > 0$. However, by construction, the Lorenz curve of the counterfactual vector $\mathbf{s_c}$ is the same as that of the final income vector $\mathbf{s_1}$ (i.e. $LC_c = LC_1$), so we have that $LC_0 \succ LC_1$ implies $W > 0$.

b) $W > 0$ implies Lorenz Worsening.

To prove the converse, namely that $W > 0$ implies $LC_0 \succ LC_1$ it is useful to consider the contrapositive version of this statement. The negation of a Lorenz-worsening can be: i) Weak Lorenz-Improvement, i.e. $LC_0 \preceq LC_1$ or ii) Lorenz curves cross. Since we ruled out Lorenz-crossings by assumption, then we only need to prove that $LC_0 \preceq LC_1$ implies $W \leq 0$. However, it is easy to see that this statement is true by reproducing the steps in part a) of this proof, now with rank-preserving equalizing transfers from richer to poorer individuals.

□

Lemma 2. Recall $\mathbf{s_c}$ is a permutation of $\mathbf{s_1}$. Since both vectors have the same distribution, the only changes are the ones due to positional swaps. If nobody changes positions $\mathbf{s_c} = \mathbf{s_1}$, and $X = 0$, trivially.

Otherwise, any positional swap will imply the transfer of resources g_{kl} from individual k to individual l where k is initially richer than l, i.e. $l < k$. Moving from $\mathbf{s_c}$ to $\mathbf{s_1}$ will imply a series of such positional swaps.

The total transfers made by individual i when going from $\mathbf{s_c}$ to $\mathbf{s_1}$ will be the sum

$$g_{i\cdot} = \sum_{j=1}^{n} g_{ij},$$

while the total transfers received by this same individual during this transition will be the sum

$$g_{\cdot i} = \sum_{j=1}^{n} g_{ji}.$$

The change in this person's income share from such transfers can be then expressed as

$$s_{1i} - s_{ci} = g_{\cdot i} - g_{i\cdot} = \sum_{j=1}^{n} g_{ji} - \sum_{j=1}^{n} g_{ij}.$$

The transfers g_{kl} appear once with a positive sign and once with a negative sign each. Furthermore, as we established before, in both cases the sender is always richer than the receiver. Hence, for each transfer g_{kl} we have that the product

$$g_{kl}s_{0l} - g_{kl}s_{0k} = g_{kl}(s_{0l} - s_{0k})$$

is negative. Hence, the term

$$X = n^{-1} \sum_{i} (s_{1i} - s_{ci}) s_{0i} = n^{-1} \sum_{i} \left(\sum_{j=1}^{n} g_{ji} - \sum_{j=1}^{n} g_{ij} \right) s_{0i}$$

will be the average of terms $g_{kl}(s_{0l} - s_{0k})$ for all transfers g_{kl}. Since all these terms are non-positive, and some will be strictly negative, then X will be negative as well. □

Proposition 2. Consider the share change regression (4)

$$\Delta s \equiv s_1 - s_0 = \gamma_s + \delta_s s_0 + u_s$$

The coefficient δ_s equals

$$\delta_s = \frac{cov(\Delta s, s_0)}{V(s_0)}.$$

Hence, its sign will be determined by the sign of the covariance

$$\begin{aligned}
cov(\Delta s, s_0) &= n^{-1}\sum_i (s_{1i} - s_{0i})s_{0i} - \overline{\Delta s} \cdot \overline{s_0} \\
&= n^{-1}\sum_i (s_{1i} - s_{0i})s_{0i} \qquad \text{(since the average share-change is zero)} \\
&= n^{-1}\sum_i [(s_{1i} - s_{ci}) + (s_{ci} - s_{0i})]s_{0i} \\
&= n^{-1}\sum_i (s_{ci} - s_{0i})s_{0i} \qquad \text{(since there are no positional changes).}
\end{aligned}$$

Hence, the sign of this covariance will equal the sign of W, and by Lemma 1, $LC_1 \prec LC_0 \iff \delta_s > 0$. □

Proposition 3. This proposition is easily proven by noting three important facts.

First, as shown in the proof of Proposition 2, the sign of the estimated coefficient δ_s in equation (4) is determined by the sign of the covariance

$$\begin{aligned}
cov(\Delta s, s_0) &= n^{-1}\sum_i (s_{1i} - s_{0i})s_{0i} \\
&= n^{-1}\sum_i [(s_{1i} - s_{ci}) + (s_{ci} - s_{0i})]s_{0i} \\
&= X + W
\end{aligned}$$

for X and W defined in (5). Hence,

$$sign(\delta_s) = sign(X + W).$$

Second, by Lemma 1, $LC_1 \succ LC_0 \iff W < 0$.

Finally, by Lemma 2, $X \leq 0$.

i)-iii) follow immediately from these facts. Also, iv) is a consequence of the aforementioned facts, since a Lorenz-improvement $LC_1 \succ LC_0$ implies $W < 0$, but share divergence requires $W > 0$. □

Lemma 3. From (1) the regression in dollars is

$$d_1 = \alpha_d + \beta_d d_0 + u_d.$$

Dividing this equation by μ_1 we obtain

$$\begin{aligned} s_1 &= \alpha_d/\mu_1 + \beta_d d_0/\mu_1 + u_d/\mu_1 \\ &= \alpha_d/\mu_1 + \beta_d[d_0/\mu_0]\mu_0/\mu_1 + u_d/\mu_1 \\ &= \alpha_d/\mu_1 + \beta_d s_0 \mu_0/\mu_1 + u_s. \end{aligned}$$

Hence,

$$\alpha_s = \alpha_d/\mu_1; \beta_s = \beta_d \mu_0/\mu_1.$$

The Lemma follows from this last equation. □

Lemma 4. Let gm denote the geometric mean of incomes at period 0, i.e.

$$gm = exp\left(n^{-1}\sum_i \ln d_i\right).$$

Let $h > 0$ be a sufficiently small rank-preserving transfer. Consider two individuals i and j such that $d_{0i} > d_{0j} > gm * exp(1)$ and assume that the only income change when going from period 0 to 1 is the transfer h between i and j.

It follows from Fields and Fei (1978) that if the transfer is equalizing it will lead to a Lorenz-improvement, and the opposite will occur if the transfer is disequalizing. The only result to establish is the sign of the coefficient δ_{log} in a log-change regression (8) under the stated conditions.

Consider the case a) of a single rank-preserving *equalizing* transfer. That is the transfer goes from the richer person i to the poorer person j. Under the stated assumptions the sign of δ_{log} will be determined by the covariance

$$cov(\Delta \ln d, \ln d_0) = n^{-1}\sum_i (\ln d_{1i} - \ln d_{0i}) \ln d_{0i} - \overline{\Delta \ln d} \cdot \overline{\ln d_0}$$

Note that all terms in the summation are zero except for i and j, so we have

$$\begin{aligned} cov(\Delta \ln d, \ln d_0) &= n^{-1}\left[(\Delta \ln d_i)\ln d_{0i} + (\Delta \ln d_j)\ln d_{0j}\right] - \overline{\Delta \ln d} \cdot \overline{\ln d_0} \\ &= n^{-1}\left[(\ln(d_{0i} - h) - \ln d_{0i})\ln d_{0i}\right] + \cdots \\ &\cdots + n^{-1}\left[(\ln(d_{0j} + h) - \ln d_{0j})\ln d_{0j}\right] - \overline{\Delta \ln d} \cdot \overline{\ln d_0} \end{aligned}$$

A First-order Taylor expansion around $h = 0$ for the first two terms is

$$n^{-1}\left[(\ln(d_{0i} - h) - \ln d_{0i}) \ln d_{0i}\right] \cong -\frac{\ln d_{0i}}{d_{0i}}\frac{h}{n}$$
$$n^{-1}\left[(\ln(d_{0j} + h) - \ln d_{0j}) \ln d_{0j}\right] \cong \frac{\ln d_{0j}}{d_{0j}}\frac{h}{n}.$$

A similar expansion for the average log-income change is

$$\overline{\Delta \ln d} \cong \frac{h}{n}\left(\frac{1}{d_{0j}} - \frac{1}{d_{0i}}\right).$$

Hence, for a marginal transfer h

$$cov(\Delta \ln d, \ln d_0) \cong \frac{h}{n}\left(\frac{\ln d_{0j} - \overline{\ln d_0}}{d_{0j}} - \frac{\ln d_{0i} - \overline{\ln d_0}}{d_{0i}}\right).$$

The sign of this covariance will be determined by the behavior of the function

$$\frac{\ln x - \overline{\ln d_0}}{x}$$

with derivative

$$\frac{1 - \ln x + \overline{\ln d_0}}{x^2}.$$

This derivative will be negative when

$$x > exp(1) * gm.$$

Hence, if individuals have income $d_{0i} > d_{0j} > exp(1) * gm$,

$$\frac{\ln d_{0j} - \overline{\ln d_0}}{d_{0j}} - \frac{\ln d_{0i} - \overline{\ln d_0}}{d_{0i}}$$

will have a positive sign and so $\delta_{log} > 0$. The case of a disequalizing transfer is proved similarly. □

Proposition 4. Rewrite the proportional change regression (9) as

$$d_1/d_0 = (\phi + 1) + \theta d_0 + e.$$

Then the sign of θ will depend on the sign of the covariance

$$\begin{aligned} cov(d_1/d_0, d_0) &= E[(d_1/d_0)d_0] - E(d_1/d_0)\mu_0 \\ &= \mu_1 - E(d_1/d_0)\mu_0. \end{aligned}$$

Hence, there will be divergence (i.e. $\theta > 0$) whenever $\mu_1 > E(d_1/d_0)\mu_0$, convergence (i.e. $\theta < 0$) whenever $\mu_1 < E(d_1/d_0)\mu_0$, otherwise the profiles will be parallel.

This condition for convergence can be re-expressed as

$$\begin{aligned} E\left(\frac{d_1}{d_0}\right)\mu_0 - \mu_1 &> 0 \\ E\left(\frac{d_1}{d_0}\right)\frac{\mu_0}{\mu_1} - 1 &> 0 \\ E\left(\frac{s_1}{s_0}\right) - 1 &> 0 \end{aligned}$$

So we can express these conditions as:
Convergence ($\theta < 0$) $\iff$ $0 < E[(s_1 - s_0)/s_0]$
Divergence ($\theta > 0$) $\iff$ $0 > E[(s_1 - s_0)/s_0]$
Parallel Profiles ($\theta = 0$) $\iff$ $0 = E[(s_1 - s_0)/s_0]$.

As we established in the proofs of Lemmas 1 and 2, and of Proposition 3, when there is a Lorenz-improvement we can go from s_0 to s_1 through a series of transfers h_{ij} and g_{kl} that:

i) appear once with a positive sign and once with a negative sign, and

ii) the sender i (or k) is always richer than the receiver j (or l)

Hence, for each transfer h_{ij} and g_{kl} we have that the products

$$\begin{aligned} \frac{h_{ij}}{s_{0j}} - \frac{h_{ij}}{s_{0i}} &= h_{ij}\left(\frac{1}{s_{0j}} - \frac{1}{s_{0i}}\right) \\ \frac{g_{kl}}{s_{0l}} - \frac{g_{kl}}{s_{0k}} &= g_{kl}\left(\frac{1}{s_{0l}} - \frac{1}{s_{0k}}\right) \end{aligned}$$

are both positive.

This in turn implies that

$$\begin{aligned} E\left(\frac{\Delta s}{s_0}\right) &= \frac{1}{n}\sum_i \frac{s_{1i}-s_{0i}}{s_{0i}} \\ &= \frac{1}{n}\sum_i \frac{(s_{1i}-s_{ci})+(s_{ci}-s_{0i})}{s_{0i}} \\ &= \frac{1}{n}\sum_i \frac{(\sum_{j=1}^n g_{ji}-\sum_{j=1}^n g_{ij})+(\sum_{j=1}^n h_{ji}-\sum_{j=1}^n h_{ij})}{s_{0i}} \end{aligned}$$

will be the sum of terms $h_{ij}(1/s_{0j} - 1/s_{0i})$ and $g_{kl}(1/s_{0l} - 1/s_{0k})$ for all the transfers h_{ij} and g_{kl}. Since all these terms are non-negative, and some will be strictly positive, then the average percentage change in shares will be positive, and by our previous derivation the coefficient θ will be negative. Namely, the exact proportional change regression will be convergent. □

Corollary 5. From the proof of Proposition 4 we know that $\theta < 0 \iff$

$$\begin{aligned} E\left(\frac{\Delta s}{s_0}\right) &= \frac{1}{n}\sum_i \frac{(s_{1i}-s_{ci})+(s_{ci}-s_{0i})}{s_{0i}} \\ &= \frac{1}{n}\sum_i \frac{(\sum_{j=1}^n g_{ji}-\sum_{j=1}^n g_{ij})+(\sum_{j=1}^n h_{ji}-\sum_{j=1}^n h_{ij})}{s_{0i}} \end{aligned}$$

is positive. The terms $h_{ij}(1/s_{0j} - 1/s_{0i})$ representing the transfers to go from $\mathbf{s_0}$ to $\mathbf{s_c}$ can be positive in the case of a Lorenz-improvement (the receiver j is poorer than the sender), or negative in the case of a Lorenz-worsening (the opposite case).

The terms $g_{kl}(1/s_{0l} - 1/s_{0k})$ representing the transfers to go from $\mathbf{s_c}$ to $\mathbf{s_1}$ are always positive because in the case of a positional swap the receiver l is always poorer than the sender k.

Hence, we can have both rising inequality (i.e. Lorenz-worsening) and convergent proportional changes as long as

$$\frac{1}{n}\sum_i \frac{s_{1i}-s_{ci}}{s_{0i}} > \left|\frac{1}{n}\sum_i \frac{s_{ci}-s_{0i}}{s_{0i}}\right|$$

□

Proposition 5. First express the regression (10) in its level-level form

$$y_1 = \alpha_y + (\delta_y + 1)y_0 + u_y.$$

Take the variance of both sides

$$V(y_1) = (\delta_y + 1)^2 V(y_0) + V(u_y).$$

Note we can rewrite the change in variances of y as

$$\Delta V(y) = V(y_1) - V(y_0) = \delta_y(\delta_y + 2)V(y_0) + V(u_y).$$

If the left-hand side of the equation is negative then it must be the case that $-2 < \delta_y < 0$. □

Corollary 7. Consider the equation for changes in the variance given at the end of the proof of Proposition 5, namely,

$$\Delta V(y) = \delta_y(\delta_y + 2)V(y_0) + V(u_y).$$

The left-hand side will be positive $\iff$

$$\frac{V(u_y)}{V(y_0)} > -\delta_y(\delta_y + 2).$$

Since $\beta_y - 1 = \delta_y$, we can rewrite this condition as

$$\frac{V(u_y)}{V(y_0)} > -(\beta_y - 1)(\beta_y + 1)$$
$$\frac{V(u_y)}{V(y_0)} > 1 - \beta_y^2.$$

□

Proposition 7. Note first that the $CV^2 = V(s)$, since

$$\begin{aligned} V(s) &= V\left(\frac{d}{\mu}\right) \\ &= \frac{1}{\mu^2}V(d) \\ &= CV^2. \end{aligned}$$

The result in Proposition 7 follows immediately by noting that by Proposition 5 a falling CV implies convergence in shares $\delta_s < 0$. Also by Proposition 6 a falling CV together with the specified type of Lorenz-crossing leads to a reduction in $I_{TS}()$.

□