Median-Based Estimation of Dynamic Panel Models with Fixed Effects

Geert Dhaene* — K.U. Leuven

Yu Zhu† — University of Wisconsin

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Abstract

We propose outlier-robust estimators for linear dynamic fixed effects panel data models where the number of observations is large and the number of time periods is small. In the simple setting of estimating the AR(1) coefficient from stationary Gaussian panel data, the estimator is (a linear transformation of) the median ratio of adjacent first-differenced data pairs. Its influence function is bounded under contamination by independent or patched additive outliers. We derive the influence function and the gross-error sensitivity explicitly. When there are independent additive outliers, the estimator is asymptotically biased towards 0, but its sign remains correct, and it has a reasonably high breakdown point. When there are patched additive outliers with point mass distribution, the asymptotic bias is upward in nearly all cases; breakdown towards 1 can occur; and the associated breakdown point increases with the patch length.

Keywords: Additive outliers, Breakdown point, Dynamic panel data, Fixed effects, Influence function, Robustness.

*Corresponding author. Address: Dept. of Economics, Naamsestraat 69, 3000 Leuven, Belgium. E-mail: geert.dhaene@econ.kuleuven.be.

†Address: Dept. of Economics, 7439 Social Science Building, 1180 Observatory Drive, Madison, WI 53706, USA. E-mail: yzhu27@wisc.edu.

1 Introduction

Suppose we have N units of observation for which there are $T \geq 3$ repeated measurements of a scalar variable y. That is, we observe y_{it} for $i = 1, ..., N$ and $t = 1, ..., T$. Assume that the y_{it} are generated by heterogeneous Gaussian AR(1) processes with common autoregressive parameter ρ, i.e.

$$y_{it} = \alpha_i + \rho y_{it-1} + \varepsilon_{it}, \qquad \varepsilon_{it} \sim N\left(0, \sigma_i^2\right), \tag{1}$$

for $i = 1, ..., N$ and $t = 2, ..., T$, with errors ε_{it} assumed to be independent across i and t, and $-1 < \rho \leq 1$. The fixed efffects α_i and the error variances $\sigma_i^2 > 0$ are nuisance parameters. The pairs (α_i, σ_i^2) may be fixed or stochastic but are assumed to be predetermined relative to y_{it} $(t = 1, ..., T)$. The initial values y_{i1} are assumed to be independent across i and drawn from the stationary distributions when $|\rho| < 1$, i.e.

$$y_{i1} \sim N\left(\frac{\alpha_i}{1-\rho}, \frac{\sigma_i^2}{1-\rho^2}\right) \tag{2}$$

for $i = 1, ..., N$, and may be any values (fixed or stochastic) when $\rho = 1$. We address the problem of estimating ρ in such a way that the estimator is robust against data contamination and also Fisher consistent as $N \to \infty$ with fixed T, absent contamination. Some generalizations of the basic model (1)–(2) are also considered.

The model $y_{it} = \alpha_i + \rho y_{it-1} + \varepsilon_{it}$ is the simplest of a range of dynamic fixed effects panel models including models with covariates and higher order dynamics. Such data and models have a long history of applied research starting with the work of Balestra and Nerlove (1966). In applications, N is often large and T small (say, $N \geq 1000$ and $T \leq 10$), so it is natural then to consider asymptotics where $N \to \infty$ while T remains fixed. As is well known, the least-squares estimator of ρ is not Fisher consistent as $N \to \infty$ with T fixed.[1] Nickell (1981) derived its large N, fixed T bias,

[1] Least-squares, here, is the within-group estimator $\sum_i \sum_{t\geq 2}(y_{it} - \overline{y}_i)(y_{it-1} - \overline{y}_{i-})/\sum_i \sum_{t\geq 2}(y_{it-1} - \overline{y}_{i-})^2$, where $\overline{y}_i$ and $\overline{y}_{i-}$ are the means of y_{it} and y_{it-1} over $t \geq 2$.

which is due to an incidental parameter problem first described, in a more general setting, by Neyman and Scott (1948). When T is small, the bias is particularly large, and the general reaction has been to move away from least-squares to generalized method of moments estimators that are large N, fixed T consistent, such as those proposed by Arellano and Bond (1991) and Blundell and Bond (1998). The latter estimators, however, were not designed to be outlier-robust. They are, in fact, highly sensitive to even a small number of outliers. Lucas, van Dijk, and Kloek (2007) show that the Arellano-Bond estimator has an unbounded influence function, and the same can be shown for the Blundell-Bond estimator.

Many authors have argued in favor of using robust statistical methods, e.g. Koenker and Bassett (1978), Huber (1981), Hampel, Ronchetti, Rousseeuw, and Stahel (1986), Rousseeuw and Leroy (1987), and Maronna, Martin, and Yohai (2006). In the panel data literature, robust inference – in the sense of robustness against contamination – has not received much attention so far. The papers by Wagenvoort and Waldmann (2002) (on bounded influence estimation of a static panel model with endogeneity but without fixed effects), Bramati and Croux (2007) (on high breakdown point estimation of the static fixed effects model), and Lucas, van Dijk, and Kloek (2007) (on bounded influence estimation of the dynamic fixed effects model) are the only ones we are aware of. Several of these authors emphasize the need for robust panel data methods on the grounds that large panels are likely to contain units i with erroneous data and that conventional methods have the tendency to mask outliers. Furthermore, the abundance of information in large panel data sets naturally suggests moving towards methods that are less fragile to outliers. The cost of sacrificing efficiency is likely to be small compared to the benefit of protection against the adverse effects of data errors.

In this paper we propose an estimator of ρ that is Fisher consistent (as $N \to \infty$, for any $T \geq 3$) and has attractive robustness properties. The basic version of the estimator is $\widehat{\rho} = 1 + 2\widehat{r}$, where $\widehat{r}$ is the median of the ratios $(y_{it} - y_{it-1})/(y_{it-1} - y_{it-2})$, and a slight variation of $\widehat{\rho}$ is exactly unbiased for any $N \geq 3$ and $T \geq 3$. Intuitively, $\widehat{\rho}$ is highly robust because it a median. It has a bounded influence function and a reasonably high breakdown point when the data are contaminated by independent or patched additive outliers. Under independent additive outlier contamination, $\widehat{\rho}$ is asymptotically biased towards zero but is sign-robust (i.e. its sign remains correct) regardless of the contamination rate and the outlier distribution. The breakdown point towards 0, for which we derive an upper and a lower bound, depends on ρ. When there are patched additive outliers with point-mass distribution, the bias of $\widehat{\rho}$ is upward except in certain cases where ρ and the contamination rate are large. We give an account of these and other robustness properties in Section 3 by applying concepts from robust time series statistics (in particular, Martin and Yohai, 1986) to the fixed effects panel data setting.

The proposed estimator is related to an estimator of the autocorrelation of a Gaussian zero-mean AR(1) process suggested by Hurwicz (1950). It can be viewed as an extension of Hurwicz' estimator to the fixed effects panel data setting accomodating non-zero means. Being a median, it also bears similarity to certain regression slope estimators under random sampling, in particular Brown and Mood's (1951) median-type GM estimator, Theil's (1950) and Sen's (1968) median of pairwise slopes, and Siegel's (1982) repeated median of pairwise slopes. Adrover and Zamar (2004) called these regression estimators "median based" and investigated their maximum asymptotic bias, concluding that that their bias-robustness properties are very satisfactory.

The normality assumption is made mainly for expository reasons. Many of the results presented below hold under the weaker assumption that the joint distribution of $y_{it} - y_{it-1}$, $t = 2, ..., T$, is elliptically contoured. We address this briefly in Section 4, together with the inclusion of covariates and higher order dynamics in (1). In Section 5, we compare $\widehat{\rho}$ with other estimators in simulations with and without data contamination. Section 6 concludes. Proofs are given in an appendix.

2 Median-based estimators

Hurwicz (1950) observed that, in the time series model $y_t = \rho y_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim N(0, \sigma^2)$, every ratio y_t/y_{t-1} is median-unbiased for ρ and conjectured that the median of those ratios is median-unbiased for ρ. Zielinski (1999) proved Hurwicz' conjecture provided that the median, say, of $x_1, ..., x_n$ with order statistics $x_{(1)} \leq ... \leq x_{(n)}$, is defined as in Zielinski (1995), viz.

$$\text{med}_Z\{x_1, ..., x_n\} = \begin{cases} x_{(k)} & \text{if } n = 2k-1, \\ Dx_{(k)} + (1-D)x_{(k+1)} & \text{if } n = 2k, \end{cases}$$

where D is a Bernoulli variate, independent of $x_1, ..., x_n$, and $\Pr[D = 0] = \Pr[D = 1] = \frac{1}{2}$. This definition differs slightly from the usual definition,

$$\text{med}\{x_1, ..., x_n\} = \begin{cases} x_{(k)} & \text{if } n = 2k-1, \\ (x_{(k)} + x_{(k+1)})/2 & \text{if } n = 2k. \end{cases}$$

The two definitions coincide when $n = 2k-1$ and are asymptotically equivalent when $x_{(\lfloor n/2 \rfloor)} - x_{(\lfloor n/2 \rfloor + 1)} = o_p(1)$ as $n \to \infty$. Thus, $\text{med}_Z\{y_t/y_{t-1}; t = 2, ..., T\}$ is median-unbiased for ρ, and the median-bias of $\text{med}\{y_t/y_{t-1}, t = 2, ..., T\}$ converges to 0 as $T \to \infty$. An attractive property of these estimators is their robustness against outliers. An additive outlier, for example, affects only two ratios, so the median ratio is almost unaffected.

Our estimators of ρ in the panel data model (1)–(2) are median-based estimators in the spirit of Hurwicz (1950). The fixed effects, α_i, are eliminated by taking first differences, $\Delta y_{it} = y_{it} - y_{it-1}$. Then the joint distribution of Δy_{it} and Δy_{it-1} is easily found as

$$\begin{pmatrix} \Delta y_{it} \\ \Delta y_{it-1} \end{pmatrix} \sim N(0, \Omega_i), \qquad \Omega_i = \frac{\sigma_i^2}{1+\rho} \begin{pmatrix} 2 & \rho - 1 \\ \rho - 1 & 2 \end{pmatrix}. \tag{3}$$

Let $r = \frac{\rho - 1}{2}$ be the correlation between Δy_{it} and Δy_{it-1}. We first derive robust estimators of r and then obtain robust estimators of ρ using $\rho = 1 + 2r$. Two types of estimators of r are proposed: the median of all ratios $\Delta y_{it}/\Delta y_{it-1}$ and the average cross-sectional median of the ratios $\Delta y_{it}/\Delta y_{it-1}$.

2.1 Median of all ratios

Because $E(\Delta y_{it}|\Delta y_{it-1}) = r\Delta y_{it-1}$, the variables $\Delta y_{it} - r\Delta y_{it-1}$ and Δy_{it-1} are uncorrelated and, by normality, independent and symmetrically distributed about zero. Therefore,

$$E\left[\text{sign}\left(\Delta y_{it} - r\Delta y_{it-1}\right)\text{sign}\Delta y_{it-1}\right] = 0$$

and so

$$E\left[\text{sign}\left(\Delta y_{it}/\Delta y_{it-1} - r\right)\right] = 0. \tag{4}$$

Hence, r can be estimated by the solution of the sample analogue of the moment condition (4),

$$\widehat{r} = \text{med}\{\Delta y_{it}/\Delta y_{it-1}; i = 1, ..., N; t = 3, ..., T\},$$

or, using Zielinski's (1995) version of the median,

$$\widehat{r}_{\text{Z}} = \text{med}_{\text{Z}}\{\Delta y_{it}/\Delta y_{it-1}; i = 1, ..., N; t = 3, ..., T\}.$$

The two estimators, $\widehat{r}$ and $\widehat{r}_{\text{Z}}$, are equal when N and T are both odd. Note that $\widehat{r}$ is the median-of-slopes estimator in the regression of Δy_{it} on Δy_{it-1}. Note also that, while each ratio $\Delta y_{it}/\Delta y_{it-1}$ is median-unbiased for r, $\widehat{r}_{\text{Z}}$ is median-unbiased for r only if $T \leq 4$ (in which case there are at most two ratios $\Delta y_{it}/\Delta y_{it-1}$ for each i) because $u_{it} = \Delta y_{it} - r\Delta y_{it-1}$ is not independent of Δy_{it-j}, $j \geq 2$. However, as $N \to \infty$, $\widehat{r}$ and $\widehat{r}_{\text{Z}}$ are asymptotically normal and centered at r.

Lemma 1 *For any $T \geq 3$, as $N \to \infty$,*

$$\sqrt{N(T-2)}\left(\widehat{r} - r\right) \xrightarrow{d} N\left(0, \frac{\pi^2(1-r^2)}{4}V_T\right),$$

$$\sqrt{N(T-2)}\left(\widehat{r}_{\text{Z}} - r\right) \xrightarrow{d} N\left(0, \frac{\pi^2(1-r^2)}{4}V_T\right),$$

where

$$V_T = \frac{1}{T-2}E\left(\sum_{t=3}^{T}\text{sign}\left(u_{it}\right)\text{sign}\left(\Delta y_{it-1}\right)\right)^2.$$

Remark 1 *In view of (3), an argument of symmetry applied to (4) yields the moment condition*

$$E\left[\operatorname{sign}\left(\Delta y_{it-1}/\Delta y_{it}-r\right)\right]=0. \tag{5}$$

We can use (4) and (5) jointly by taking the median of all ratios $\Delta y_{it}/\Delta y_{it-1}$ *and their reciprocals,* $\Delta y_{it-1}/\Delta y_{it}$. *Let* $\widetilde{r}$ *and* $\widetilde{r}_{\mathrm{Z}}$ *be the resulting estimators, using* $\operatorname{med}\{\cdot\}$ *and* $\operatorname{med}_{\mathrm{Z}}\{\cdot\}$, *respectively. By an argument similar to that used in the proof of Lemma 1,*

$$\sqrt{N\left(T-2\right)}\left(\widetilde{r}-r\right)\xrightarrow{d}N\left(0,\frac{\pi^2(1-r^2)}{4}W_T\right),$$

and similarly for $\widetilde{r}_{\mathrm{Z}}$, *where*

$$W_T=\frac{1}{4\left(T-2\right)}E\left(\sum_{t=3}^{T}\operatorname{sign}\left(u_{it}\right)\operatorname{sign}\left(\Delta y_{it-1}\right)+\sum_{t=3}^{T}\operatorname{sign}\left(v_{it-1}\right)\operatorname{sign}\left(\Delta y_{it}\right)\right)^2$$

and $v_{it-1}=\Delta y_{it-1}-r\Delta y_{it}$. *Because the variables* $\sum_{t=3}^{T}\operatorname{sign}\left(u_{it}\right)\operatorname{sign}\left(\Delta y_{it-1}\right)$ *and* $\sum_{t=3}^{T}\operatorname{sign}\left(v_{it-1}\right)\operatorname{sign}\left(\Delta y_{it}\right)$ *have the same distribution and are not perfectly correlated,* $W_T<V_T$ *for all* $T\geq 3$.

Remark 2 *If* $T\to\infty$ *with* N *fixed or* $N\to\infty$, *the limit distributions of* $\widehat{r}$, $\widehat{r}_{\mathrm{Z}}$, $\widetilde{r}$, *and* $\widetilde{r}_{\mathrm{Z}}$ *are obtained from those stated above by replacing* V_T *and* W_T *by* V_∞ *and* W_∞. *This includes the time series setting, where* $N=1$.

We can estimate $\rho=1+2r$ by $\widehat{\rho}=1+2\widehat{r}$ or by $\widehat{\rho}_{\mathrm{Z}}$, $\widetilde{\rho}$, or $\widetilde{\rho}_{\mathrm{Z}}$ with obvious definitions. The limit distributions are given by

$$\sqrt{N\left(T-2\right)}\left(\widehat{\rho}-\rho\right)\xrightarrow{d}N\left(0,\pi^2(1-r^2)V_T\right),$$
$$\sqrt{N\left(T-2\right)}\left(\widetilde{\rho}-\rho\right)\xrightarrow{d}N\left(0,\pi^2(1-r^2)W_T\right),$$

and similarly for $\widehat{\rho}_{\mathrm{Z}}$ and $\widetilde{\rho}_{\mathrm{Z}}$. Any of these estimators may be replaced by the nearest boundary of $[-1,1]$ if it falls outside this interval.

2.2 Average cross-sectional median of ratios

A slight variation of the above estimators of r, and hence of ρ, gives estimators that are exactly unbiased and whose limit distributions are unchanged.

Start by noting that $\Delta y_{it}/\Delta y_{it-1} = (\Delta y_{it} - r\Delta y_{it-1})/\Delta y_{it-1} + r$ is a Cauchy variate with location r and scale $(1-r^2)^{1/2}$. Hence, $\Delta y_{it}/\Delta y_{it-1}$ is symmetrically distributed around r, as is its cross-sectional median,

$$\widehat{r}_t = \text{med}\{\Delta y_{it}/\Delta y_{it-1}; i = 1, ..., N\},$$

for each $t \geq 3$. Assuming $N \geq 3$, it follows that $E(\widehat{r}_t) = r$ and $E(\widehat{r}_.) = r$, where $\widehat{r}_. = (T-2)^{-1}\sum_{t=3}^{T}\widehat{r}_t$. The same holds if $\text{med}_{\text{Z}}\{\cdot\}$ is used in the definition of $\widehat{r}_t$ instead of $\text{med}\{\cdot\}$. The following lemma shows that $\widehat{r}_.$ and $\widehat{r}$ have the same limit distribution.

Lemma 2 *For any $T \geq 3$, as $N \to \infty$,*

$$\sqrt{N(T-2)}\,(\widehat{r}_. - r) \xrightarrow{d} N\left(0, \frac{\pi^2(1-r^2)}{4}V_T\right)$$

with V_T as in Lemma 1. The same holds if $T \to \infty$ with N fixed or $N \to \infty$, on replacing V_T with V_∞.

Remark 3 *Let $\widetilde{r}_t$ be the median of $\{\Delta y_{it}/\Delta y_{it-1}, \Delta y_{it-1}/\Delta y_{it}; i = 1, ..., N\}$, using* $\text{med}\{\cdot\}$ *or* $\text{med}_{\text{Z}}\{\cdot\}$*, and let $\widetilde{r}_. = (T-2)^{-1}\sum_{t=3}^{T}\widetilde{r}_t$. If $N \geq 3$ and* $\text{med}_{\text{Z}}\{\cdot\}$ *is used, $E(\widetilde{r}_t) = E(\widetilde{r}_.) = r$. Whichever version of the median is used, for any $T \geq 3$, as $N \to \infty$,*

$$\sqrt{N(T-2)}\,(\widetilde{r}_. - r) \xrightarrow{d} N\left(0, \frac{\pi^2(1-r^2)}{4}W_T\right),$$

with W_T as before. Note that for this limit result to hold for fixed N and $T \to \infty$, $\text{med}_{\text{Z}}\{\cdot\}$ *must be used in the definition of $\widetilde{r}_t$ because otherwise $E(\widetilde{r}_t) = E(\widetilde{r}_.) \neq r$ due to pairwise dependencies among the elements of $\{\Delta y_{it}/\Delta y_{it-1}, \Delta y_{it-1}/\Delta y_{it}; i = 1, ..., N\}$.*

The unbiased estimators $\widehat{r}_.$ and $\widetilde{r}_.$ for r induce unbiased estimators $\widehat{\rho}_. = 1 + 2\widehat{r}_.$ and $\widetilde{\rho}_. = 1 + 2\widetilde{r}_.$ for ρ (for any $T \geq 3$ and $N \geq 3$). The limit distributions of $\widehat{\rho}_.$ and $\widetilde{\rho}_.$ are the same as those of $\widehat{\rho}$ and $\widetilde{\rho}$, respectively.

3 Robustness properties

We study the robustness of the proposed estimators in a setting where the errors are homoskedastic ($\sigma_i^2 = \sigma^2$) and the observed data are subject to contamination by additive outliers (AO) that are independent across i and, across t, are independent or occur in patches.

The results on asymptotic bias and the robustness measures derived from it are presented with reference to the fixed effects panel data setting. Thus, the asymptotics involve $N \to \infty$ and any $T \geq 3$. However, the asymptotic bias is shown to be independent of T, hence for $\widehat{\rho}$ and $\widetilde{\rho}$, it coincides with the asymptotic bias as $T \to \infty$ for any $N \geq 1$. Therefore, with minor notational modification, the results for $\widehat{\rho}$ and $\widetilde{\rho}$ apply to the time series context as well.

3.1 Independent AO with point-mass distribution

Suppose the observed data are subject to AO contamination in the sense of Fox (1972) type I outliers. Specifically, we observe

$$y_{it}^{\zeta,\varepsilon} = y_{it} + a_{it}, \qquad \Pr[a_{it} = \zeta] = 1 - \Pr[a_{it} = 0] = \varepsilon,$$

where a_{it} is independent across i and t, and independent of the uncontaminated data, y_{it} $(i = 1, ..., N; t = 1, ..., T)$. Thus, ε is the fraction of contaminated observations and ζ is the size of the AO.

Let $\widehat{\varrho}$ be an estimator of ρ, based on $y_{it}^{\zeta,\varepsilon}$ $(i = 1, ..., N; t = 1, ..., T)$, and let $\varrho(\rho, \zeta, \varepsilon, T)$ be its probability limit as $N \to \infty$ and T remains fixed. Assume $\varrho(\rho, \zeta, 0, T) = \rho$, i.e. $\widehat{\varrho}$ is Fisher consistent absent contamination. The robustness of $\widehat{\varrho}$ against a non-negligible fraction of AO can then be measured by its asymptotic bias,

$$\text{Bias}(\widehat{\varrho}; \rho, \zeta, \varepsilon, T) = \varrho(\rho, \zeta, \varepsilon, T) - \rho,$$

by the maximum bias,

$$\text{MB}(\widehat{\varrho}; \rho, \varepsilon, T) = \text{Bias}(\widehat{\varrho}; \rho, \zeta^*, \varepsilon, T), \quad \text{where } \zeta^* = \arg\sup_{\zeta} |\text{Bias}(\widehat{\varrho}; \rho, \zeta, \varepsilon, T)|,$$

and by the breakdown point towards $c \neq \rho$, which we define as

$$\mathrm{BP}_c(\widehat{\varrho}; \rho, T) = \sup_{\varepsilon}\{\varepsilon : \sup_{\zeta} \varrho(\rho, \zeta, \varepsilon, T) < c \text{ if } \rho < c; \ \inf_{\zeta} \varrho(\rho, \zeta, \varepsilon, T) > c \text{ if } \rho > c\}.$$

Intuitively, at least a fraction $\mathrm{BP}_c(\widehat{\varrho}; \rho, T)$ of contamination is needed to drive $\widehat{\varrho}$ (asymptotically) to or beyond c. Of specific interest is the breakdown point towards a boundary point of the parameter space, ± 1, and towards 0.[2] The effect of a small amount of contamination can be measured by the influence function,

$$\mathrm{IF}(\widehat{\varrho}; \rho, \zeta, T) = \lim_{\varepsilon \downarrow 0} \frac{\varrho(\rho, \zeta, \varepsilon, T) - \rho}{\varepsilon} = \left.\frac{\partial \mathrm{Bias}(\widehat{\varrho}; \rho, \zeta, \varepsilon, T)}{\partial \varepsilon}\right|_{\varepsilon=0},$$

and by the gross-error sensitivity,

$$\mathrm{GES}(\widehat{\varrho}; \rho, T) = \mathrm{IF}(\widehat{\varrho}; \rho, \zeta^{**}, T), \quad \text{where } \zeta^{**} = \arg\sup_{\zeta} |\mathrm{IF}(\widehat{\varrho}; \rho, \zeta, T)|.$$

Consider $\widehat{r} = \mathrm{med}\{\Delta y_{it}^{\zeta,\varepsilon} / \Delta y_{it-1}^{\zeta,\varepsilon}; i = 1, ..., N; t = 3, ..., T\}$ and $\widehat{\rho} = 1 + 2\widehat{r}$. Clearly, $\mathrm{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon, T) = 2b$ where b, defined as the asymptotic bias of $\widehat{r}$ as an estimator of r, solves

$$\Pr\left[\frac{\Delta y_{it}^{\zeta,\varepsilon} - r\Delta y_{it-1}^{\zeta,\varepsilon}}{\Delta y_{it-1}^{\zeta,\varepsilon}} \leq b\right] = \frac{1}{2} \tag{6}$$

for given $r = \frac{\rho-1}{2}$, ζ, and ε. Note that b does not depend on T, hence $\mathrm{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon, T) = \mathrm{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon)$, and similarly for MB, BP_c, IF, and GES. Furthermore, by the assumption that the AO are independent and because $\Delta y_{it}^{\zeta,\varepsilon} / \Delta y_{it-1}^{\zeta,\varepsilon}$ and $\Delta y_{it-1}^{\zeta,\varepsilon} / \Delta y_{it}^{\zeta,\varepsilon}$ have the same distribution, all of $\widehat{\rho}$, $\widetilde{\rho}$, $\widehat{\rho}_{\cdot}$, and $\widetilde{\rho}_{\cdot}$ have the same Bias, MB, and so on. In the appendix, we show how to compute $\mathrm{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon)$ and $\mathrm{MB}(\widehat{\rho}; \rho, \varepsilon)$, and prove the following.

[2] For recent discussions on the definition of breakdown, see Genton and Lucas (2003) and Gather and Davies (2005).

Theorem 1 *Under independent AO contamination occurring with probability* $\varepsilon > 0$ *and point-mass distribution at* $\zeta \neq 0$*,*
(i) $\text{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon) = \text{Bias}(\widehat{\rho}; \rho, \zeta, 1-\varepsilon) = \text{Bias}(\widehat{\rho}; \rho, -\zeta, \varepsilon)$*;*
(ii) $|\text{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon)|$ *is increasing in* $\varepsilon(1-\varepsilon)$ *for all* $\rho \neq 0$*;*
(iii) $\text{sign}(\text{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon)) = -\text{sign}\rho$*;*
(iv) $0 \leq |\text{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon)| \leq |\rho|$*, with equalities if and only if* $\rho = 0$*;*
(v) $|\text{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon)|$ *is increasing in* ρ *on* $[0, 1]$*.*

Hence, the worst contamination rate is $\varepsilon = \frac{1}{2}$, and the asymptotic bias is towards zero. Note that $\text{BP}_1(\widehat{\rho}; \rho) = \text{BP}_{-1}(\widehat{\rho}; \rho) = 1$. The four leftmost plots of Figure 1 graph $\text{Bias}(\widehat{\rho}; \cdot, \zeta, \varepsilon)$ for $\zeta/\sigma = 0.5, 1, 2, 10$ and $\varepsilon = 0.05, 0.1, 0.2, 0.5$. For each ζ/σ, the curves are further away from the zero bias line as ε increases to 0.5.[3] For all values of ε and ζ, the largest bias occurs at $\rho = 1$. The righmost plot of Figure 1 gives maximum bias, $\text{MB}(\widehat{\rho}; \cdot, \varepsilon)$, for $\varepsilon = 0.05, 0.1, 0.2, 0.5$ (again the curves are further away from the zero bias line as ε increases). Although the maximum bias can be substantial, it never alters the sign of the estimator (asymptotically). Thus, $\widehat{\rho}$ is sign-robust under the type of contamination considered. Equivalently, $\text{BP}_0(\widehat{\rho}; \rho) = 1$.

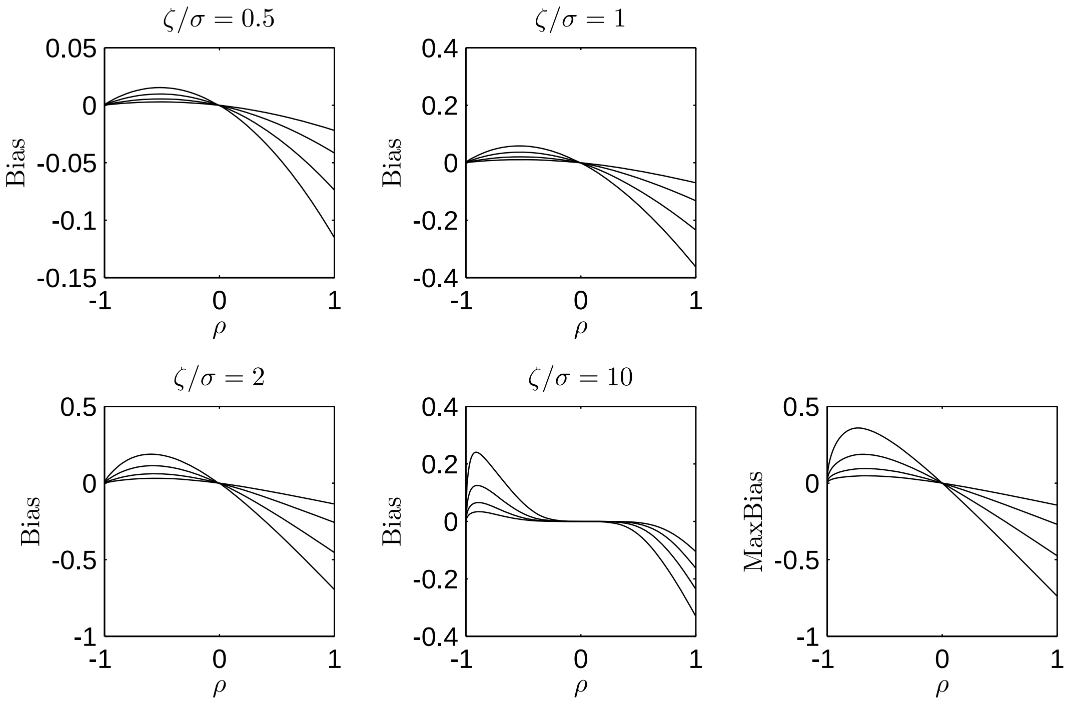
Theorem 2 *Under independent AO contamination with point-mass distribution at* $\zeta \neq 0$*,*

$$\begin{aligned} \text{IF}(\widehat{\rho}; \rho, \zeta) &= -2\pi\sqrt{1-r^2}\left[\Phi\left(\frac{\sqrt{1+r}\zeta}{\sigma}\right) - \Phi\left(\frac{-\sqrt{1+r}\zeta}{\sigma}\right)\right] \\ &\quad \times \left[\Phi\left(\frac{(1+r)\zeta}{\sigma\sqrt{1-r}}\right) - \Phi\left(\frac{-r\zeta}{\sigma\sqrt{1-r}}\right)\right]. \end{aligned}$$

Clearly, $\text{IF}(\widehat{\rho}; \rho, \zeta)$ is bounded. The bracketed factors are bounded (in absolute value) by 1 and $\frac{1}{2}$, respectively, so $|\text{IF}(\widehat{\rho}; \rho, \zeta)| \leq \pi$. For all values of ζ/σ, $|\text{IF}(\widehat{\rho}; \rho, \zeta)|$ is maximum when $\rho = 1$. Further, $\text{sign}(\text{IF}(\widehat{\rho}; \rho, \zeta)) = -\text{sign}\rho$, $\text{IF}(\widehat{\rho}; 0, \zeta) = 0$, $\lim_{\rho \downarrow -1} \text{IF}(\widehat{\rho}; \rho, \zeta) = 0$, and $\lim_{\zeta \to \pm\infty} \text{IF}(\widehat{\rho}; 1, \zeta) = -\pi$.

[3] Somewhat unconventionally, we plot Bias, MB, IF, and GES against ρ for various levels of ε and ζ/σ (where applicable) because this gives a better view on how contamination affects $\widehat{\rho}$ across the range of values of ρ and enhances comparability across curves.

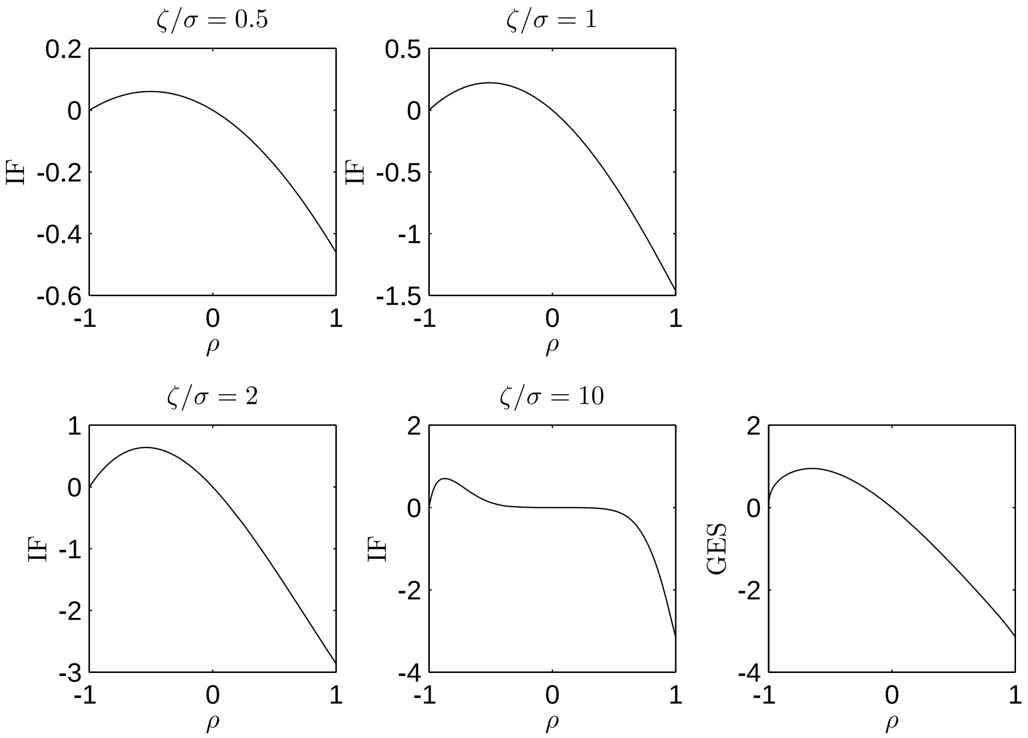
Figure 1: Bias and maximum bias under independent AO



AO with point mass at ζ and contamination rate $\varepsilon = .05, .1, .2, .5$.
As ε increases, the corresponding curve is steeper.

Figure 2 shows $\mathrm{IF}(\widehat{\rho}; \cdot, \zeta)$ for $\zeta/\sigma = 0.5, 1, 2, 10$, and $\mathrm{GES}(\widehat{\rho}; \cdot)$. Comparing the IF and GES curves with the Bias and MB curves for large values of ε, we see that they have similar patterns, but over most of the range of ρ the approximations εIF and εGES overestimate Bias and MB (in absolute value); in fact, $\varepsilon(1-\varepsilon)$IF and $\varepsilon(1-\varepsilon)$GES are somewhat better approximations to Bias and MB.

Figure 2: IF and GES under independent AO



AO with point mass at ζ.

3.2 Independent AO with arbitrary distribution

The asymptotic bias of $\widehat{\rho}$ can be also calculated under independent AO with an arbitrary distribution instead of a point-mass distribution. Let the observed data be $y_{it}^{\zeta,\varepsilon} = y_{it} + a_{it}$, where

$$\Pr[a_{it} \neq 0] = \varepsilon, \qquad \Pr[a_{it} \leq z | a_{it} \neq 0] = G_\zeta(z),$$

and G_ζ is the cdf of ζ. (As before, a_{it} is assumed independent across i and t, and independent of y_{it}.) Then the asymptotic bias of $\widehat{\rho}$ is $\text{Bias}(\widehat{\rho}; \rho, G_\zeta, \varepsilon) = 2b$ where b solves (6). Solving this equation for given but arbitrary G_ζ is essentially the same as when G_ζ is degenerate. Details are given in the appendix. The following robustness properties of $\widehat{\rho}$ continue to hold: $\widehat{\rho}$ is biased towards 0, is sign-robust, has a bounded influence function, and cannot break down to 1 or -1. Breakdown to 0, however, can occur when ε is sufficiently large.

Theorem 3 *Under independent AO contamination occurring with probability $\varepsilon > 0$ and distribution G_ζ,*
(i) $\text{sign}(\text{Bias}(\widehat{\rho}; \rho, G_\zeta, \varepsilon)) = -\text{sign}\rho$*;*
(ii) $0 \leq |\text{Bias}(\widehat{\rho}; \rho, G_\zeta, \varepsilon)| \leq |\rho|$*, with equalities if and only if* $\rho = 0$.

Thus, under independent outlier contamination, for any G_ζ and $\varepsilon > 0$, it holds that $-\rho < \text{Bias}(\widehat{\rho}; \rho, G_\zeta, \varepsilon) < 0$ if $\rho > 0$, and $-\rho > \text{Bias}(\widehat{\rho}; \rho, G_\zeta, \varepsilon) > 0$ if $\rho < 0$. It follows that the maximum bias of $\widehat{\rho}$, over all G_ζ, cannot exceed ρ, i.e. $|\text{MB}(\widehat{\rho}; \rho, \varepsilon)| \leq |\rho|$.

We have two results regarding breakdown to 0. The first is a characterization of breakdown under symmetric four-point AO distributions. The second is a lower bound to the breakdown point under AO with arbitrary distributions. Together, the two results bound the breakdown point under AO with arbitrary distributions from above and below.

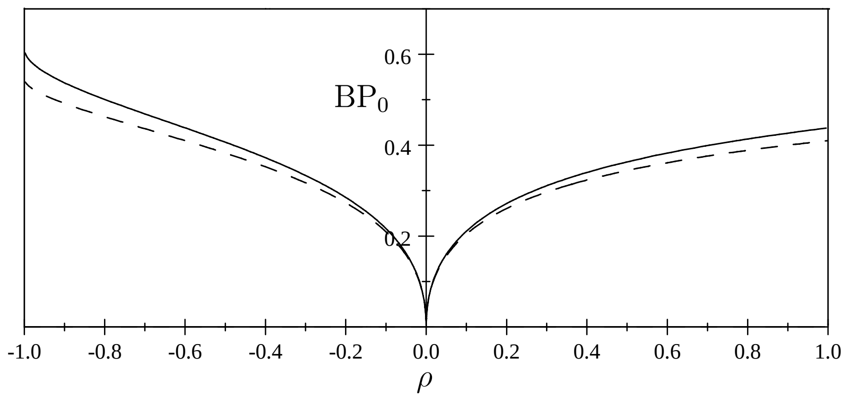
Theorem 4 *Let there be independent AO contamination occurring with probability ε, where $0 < \varepsilon < 1$, and symmetric four-point distribution G_ζ defined by* $\Pr[\zeta = z] = \Pr[\zeta = -z] = \Pr[\zeta = 2z] = \Pr[\zeta = -2z] = \frac{1}{4}$. *Then* $\text{Bias}(\widehat{\rho}; \rho, G_\zeta, \varepsilon) \to -\rho$ *as $z \to \infty$ if and only if*

$$\left(\frac{\varepsilon}{4(1-\varepsilon)} + 4\left(\frac{1-\varepsilon}{\varepsilon}\right)^2 \right) \frac{1}{\pi} \arctan \frac{\left|r + \frac{1}{2}\right|}{\sqrt{1-r^2}} \leq 1. \tag{7}$$

Under independent AO contamination with arbitrary symmetric four-point distributions, $\text{BP}_0(\widehat{\rho}; \rho)$ *is the smallest ε that solves (7) with equality.*

The solid curve in Figure 3 shows the breakdown point to 0, as a function ρ, under AO with symmetric four-point distributions. Obviously, this breakdown point is an upper bound to the breakdown point to 0 under AO with arbitrary distributions. We conjecture that this bound is sharp, i.e., that $\mathrm{BP}_0(\widehat{\rho};\rho)$ under AO with arbitrary distributions is the smallest ε that solves (7) with equality. We were not able to prove this conjecture but we derived a lower bound to $\mathrm{BP}_0(\widehat{\rho};\rho)$ that is close to the conjectured $\mathrm{BP}_0(\widehat{\rho};\rho)$. This lower bound is given below and shown by the dashed curve in Figure 3.

Figure 3: Breakdown point to 0 under independent AO



Solid curve: $\mathrm{BP}_0(\widehat{\rho};\rho)$ under symmetric four-point AO.
Dashed curve: lower bound to $\mathrm{BP}_0(\widehat{\rho};\rho)$ under arbitrary AO.

Theorem 5 *Under arbitrary independent AO contamination,* $\mathrm{BP}_0(\widehat{\rho};\rho)$ *is at least as large as the smallest* $\varepsilon > 0$ *solving*

$$100\left(\frac{1-\varepsilon}{25-17\varepsilon}\right)\left(\frac{1-\varepsilon}{\varepsilon}\right)^2\frac{1}{\pi}\arctan\frac{\left|r+\frac{1}{2}\right|}{\sqrt{1-r^2}}=1. \tag{8}$$

The influence function follows straightforwardly from Theorem 2 on integrating over the distribution of ζ, and the gross-error sensitivity is as in the degenerate case.

3.3 Patches of additive outliers with point-mass distribution

Here we investigate the robustness of the proposed estimators under patches of additive outliers. Following Martin and Yohai (1986), we define patched

AOs with patch length $k \geq 2$ and contamination rate ε using an auxiliary Bernoulli process that marks the onset of patches. Let z_{it}^{p} be a Bernoulli(p) variate, independent across i and t, and let the contaminated data be

$$y_{it}^{\zeta,\varepsilon,k} = \begin{cases} y_{it} + a_{it} & \text{if } z_{it-l}^{p} = 1 \text{ for some } l = 0, 1, \ldots .k-1, \\ y_{it} & \text{otherwise,} \end{cases}$$

where y_{it} is uncontaminated, $\Pr[a_{it} = \zeta] = 1$, and p satisfies $(1-p)^k = 1-\varepsilon$. Let $\widehat{\varrho}$ be an estimator of ρ, based on $y_{it}^{\zeta,\varepsilon,k}$, and let $\varrho(\rho, \zeta, \varepsilon, k, T)$ be its probability limit as $N \to \infty$ and T is fixed. The asymptotic bias and other robustness measures derived from it are defined as in Section 4.1 with the inclusion of k as an additional argument. We give a method to compute $\text{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon, k, T) = \text{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon, k)$ in the appendix. Again, $\widehat{\rho}$, $\widetilde{\rho}$, $\widehat{\rho}_{\cdot}$, and $\widetilde{\rho}_{\cdot}$ all have the same asymptotic bias.

Under patched AO, we no longer have the property that $\widehat{\rho}$ is always biased towards 0.

Theorem 6 *Under patched AO contamination occurring with probability $\varepsilon > 0$, point-mass distribution at $\zeta \neq 0$, and patch length $k \geq 2$,*

$$\text{sign}\left(\text{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon, k)\right) = \begin{cases} 1 & \text{if } \rho \leq 0, \\ \text{determined by } \rho,\ \zeta,\ k, \text{ and } \varepsilon & \text{if } \rho > 0. \end{cases}$$

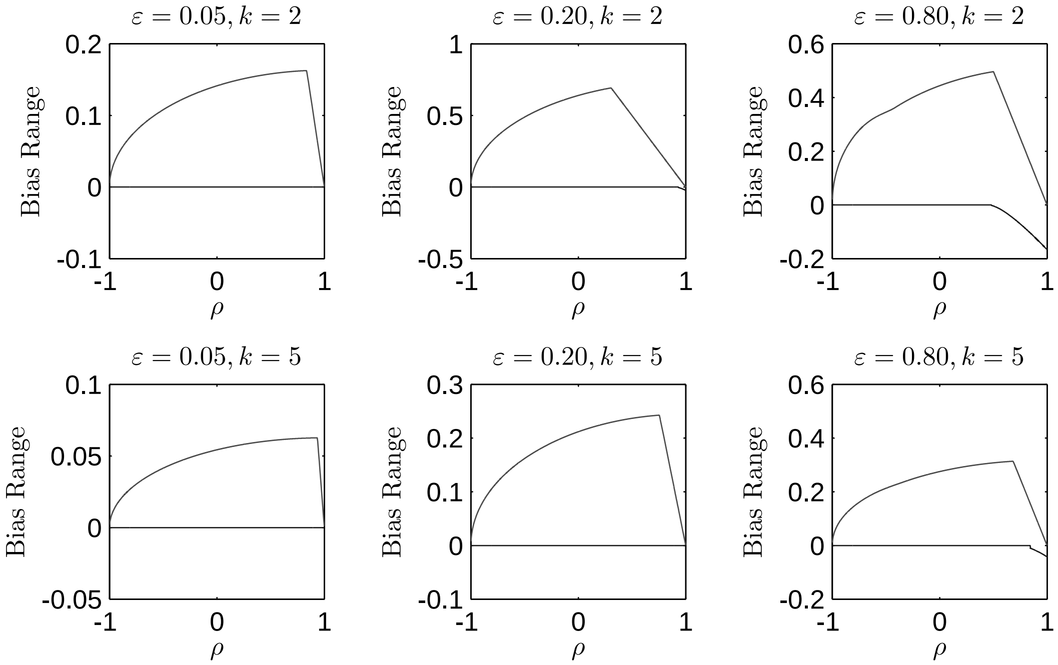
If $\rho > 0$, while $\widehat{\rho}$ can be biased in either direction, it is upward biased for most values of ρ, ζ, k, and ε. This can be inferred from the bias range of $\widehat{\rho}$, defined as the interval

$$\text{BR}(\widehat{\rho}; \rho, \varepsilon, k) = [\inf_{\zeta} \text{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon, k), \sup_{\zeta} \text{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon, k)].$$

Figure 4 displays $\text{BR}(\widehat{\rho}; \rho, \varepsilon, k)$ as a function of ρ for $\varepsilon = 0.05, 0.2, 0.8$ and $k = 2, 5$. Only when ρ is large enough and ε is very large is there a possibility that $\widehat{\rho}$ is downward biased.

Unlike in the case of independent AO, where the influence function and ρ have opposite signs, under patched AO the influence function is always positive, except at $\rho = 1$, where it is 0.

Figure 4: Bias range under patched AO



AO with patch length k and contamination rate ε.

Top and bottom curves are the bounds of the bias range.

Theorem 7 *Under patched AO contamination with point-mass distribution at $\zeta \neq 0$ and patch length $k \geq 2$,*

$$\begin{aligned}\mathrm{IF}(\widehat{\rho}; \rho, \zeta, k) &= -\frac{2\pi}{k}\sqrt{1-r^2}\left[\Phi\left(\frac{\sqrt{1+r}\zeta}{\sigma}\right) - \Phi\left(\frac{-\sqrt{1+r}\zeta}{\sigma}\right)\right] \\ &\quad \times \left[\Phi\left(\frac{r\zeta}{\sigma\sqrt{1-r}}\right) - \Phi\left(\frac{-r\zeta}{\sigma\sqrt{1-r}}\right)\right]\end{aligned}$$

and

$$\mathrm{GES}(\widehat{\rho}; \rho, k) = \begin{cases} \frac{2\pi}{k}\sqrt{1-r^2} & \text{if } \rho < 1, \\ 0 & \text{if } \rho = 1. \end{cases}$$

Because the bias is always upward when $\rho \leq 0$, $\widehat{\rho}$ cannot break down to -1. Breakdown to 1 is possible, as $|\zeta| \to \infty$, if ρ satisfies a certain condition.

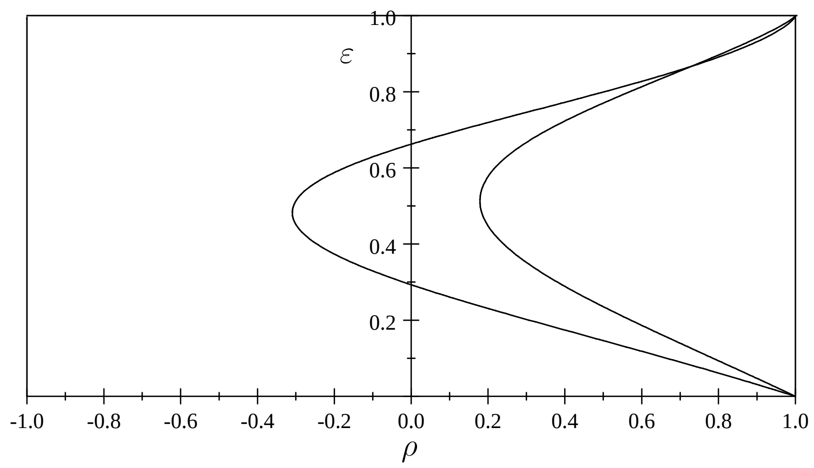
Theorem 8 *Let $\rho < 1$. Under patched AO contamination occurring with probability $\varepsilon > 0$, point-mass distribution at ζ, and patch length $k \geq 2$, $\widehat{\rho}$ breaks down to 1 as $|\zeta| \to \infty$ if and only if*

$$\left(\frac{1}{p\,(1-p)^{k+1}} - \frac{4-p}{1-p} \right) \frac{1}{\pi} \arctan \frac{-r}{\sqrt{1-r^2}} \leq 1,$$

where p is defined by $(1-p)^k = 1 - \varepsilon$.

The region (ρ, ε) where, under patched AO with patch length $k = 2, 3$, $\widehat{\rho}$ is asymptotically driven to 1 as $|\zeta| \to \infty$, is shown in Figure 5. The breakdown region lies between the curve (left curve for $k = 2$; right curve for $k = 3$) and the vertical line at $\rho = 1$. When $k = 2$, breakdown to 1 requires ρ to satisfy $\frac{1}{\pi} \arctan \frac{-r}{\sqrt{1-r^2}} \leq \frac{p(1-p)^3}{1-p(p-4)(1-p)^2}$, which can only occur if $\rho \geq -0.3093$. When $k = 3$, breakdown to 1 can only occur if $\rho \geq 0.1789$. As k increases, the region of breakdown to 1 shrinks. Intuitively, this is because $\widehat{\rho}$ depends only on the differences Δy_{it}, so if ε is kept fixed while k increases, the number of affected differences decreases and the estimator is less influenced. This also shows up in the influence function.

Figure 5: Breakdown region towards 1 under patched AO



Left curve: patch length $k = 2$. Right curve: $k = 3$.
Breakdown occurs, as $|\zeta| \to \infty$, between the curve and the line $\rho = 1$.

3.4 Patches of additive outliers with arbitrary distribution

As in the case of independent AO, we can also allow patched AO to be drawn from an arbitrary distribution G_ζ (which is not degenerate at 0). We still use the auxilliary Bernoulli process z_{it}^p defined in the previous subsection to mark the onset of a patch of length $k \geq 2$. Once a patch of AO sets off, say $z_{it}^p = 1$, ζ_{it} is drawn from G_ζ and the patch of AO is $a_{it+l} = \zeta_{it}$ for $l = 0, ..., \min\{k-1, n\}$, where n is the smallest positive integer for which $z_{it+n}^p = 1$. Thus, when two or more patches overlap, a new ζ is drawn at the onset of each new patch and replaces the previous value.[4] The contaminated data, then, are

$$y_{it}^{G_\zeta,\varepsilon,k} = \begin{cases} y_{it} + a_{it}, & \text{if } z_{it-l}^p = 1 \text{ for some } l = 0, 1, ...k-1, \\ y_{it}, & \text{otherwise,} \end{cases}$$

with a_{it} as described and $(1-p)^k = 1 - \varepsilon$. The appendix outlines how to compute $\text{Bias}(\widehat{\rho}; \rho, G_\zeta, \varepsilon, k, T) = \text{Bias}(\widehat{\rho}; \rho, G_\zeta, \varepsilon, k)$ under this type of contamination. We have the following result.

Theorem 9 *Under patched AO contamination occurring with probability $\varepsilon > 0$, distribution G_ζ, and patch length $k \geq 2$,*

$$\text{sign}\left(\text{Bias}(\widehat{\rho}; \rho, G_\zeta, k, \varepsilon)\right) = \begin{cases} 1 & \text{if } \rho \leq 0, \\ \text{determined by } \rho,\ G_\zeta,\ k, \text{ and } \varepsilon & \text{if } \rho > 0. \end{cases}$$

4 Extensions and remarks

4.1 Median-based moments: further examples

Median-based moment conditions can also be derived for more general dynamic fixed effects models. Here we give a few examples and a brief discussion of robustness issues, without attempt at generality. The models are

[4] Alternatively, and perhaps more naturally as a model of additive outliers, one could set $a_{it} = \sum_{l=0}^{k-1} I(z_{it-l}^p = 1)\zeta_{it-l}$. However, bias computations are more complicated under this specification, especially when k is large. Furthermore, this definition is not compatible with the definition of patched outliers with point-mass distribution given by Martin and Yohai (1986) and employed in the previous subsection.

restrictive, and the list of displayed moment conditions is not exhaustive. Because the estimands are overidentified by median-based moment conditions, the question arises of how to select moment conditions given concerns for robustness and efficiency. We do not formally address this question here, although our selection of moment conditions below was guided by robustness and efficiency considerations and also by simplicity, thus allowing some properties to be stated without further analysis.

Higher-order AR. Consider the AR(2) panel data model

$$y_{it} = \alpha_i + \rho_1 y_{it-1} + \rho_2 y_{it-2} + \varepsilon_{it}, \qquad \varepsilon_{it} \sim N\left(0, \sigma_i^2\right).$$

Here ρ_1 and ρ_2 are identified, for example, by

$$\mathrm{Med}\left(\frac{y_{it} - y_{it-2}}{y_{it-1} - y_{it-3}}\right) = \frac{\rho_1}{2}, \tag{9}$$

$$\mathrm{Med}\left(\frac{y_{it} - y_{it-1}}{y_{it-1} - y_{it-2}}\right) = \frac{\rho_1 - \rho_2 - 1}{2}, \tag{10}$$

where Med(·) denotes the population median. Median-based estimators follow from the sample medians. Because $\rho_1 - \rho_2$ plays the same role in (10) as ρ in the AR(1) model in (4), the corresponding estimator of $\rho_1 - \rho_2$ inherits the robustness properties established earlier for $\widehat{\rho}$ in the AR(1) model. Condition (9) is similar to (10) but involves four consecutive observations instead of three. The robustness properties of the corresponding estimator of ρ_1 can be analyzed along similar lines as before. It is easy to see that the estimator has a positive breakdown point. Alternatively, estimation can be based on (9) and

$$\mathrm{Med}\left(\frac{\Delta y_{it} - \rho_1 \Delta y_{it-1}}{\Delta y_{it-2}}\right) = \rho_2, \tag{11}$$

yielding a different estimator of ρ_2. Simulations suggest that those two estimators of ρ_2 cannot be ranked in terms of efficiency or robustness.

Covariates. Consider

$$y_{it} = \alpha_i + \beta x_{it} + \rho y_{it-1} + \varepsilon_{it}, \qquad \varepsilon_{it} \sim N\left(0, \sigma_i^2\right), \qquad x_{it} \sim N\left(\mu_i, \sigma_{xi}^2\right), \tag{12}$$

where x_{it} is independent across i and t. Here ρ and β are identified by

$$\text{Med}\left(\frac{\Delta y_{it}}{\Delta y_{it-1}}\right) = \frac{\rho - 1}{2}, \tag{13}$$

$$\text{Med}\left(\frac{\Delta y_{it} - \rho \Delta y_{it-1}}{\Delta x_{it}}\right) = \beta. \tag{14}$$

The corresponding estimator of ρ has the same robustness properties as $\widehat{\rho}$ in the pure AR(1), even when x_{it} is subject to independent AO contamination. Given the estimate $\widehat{\rho}$ of ρ, the sample median associated with (14) yields a median-of-slopes estimator of β.[5] Under independent AO contamination affecting (y_{it}, x_{it}), the estimator of β has a breakdown point (defined, as usual, as the minimum contamination rate required to carry the estimator over any bound) of at least $1 - \sqrt[3]{0.5} = 0.206$. To see this, fix $\widehat{\rho}$ and note that breakdown requires half of the ratios $\frac{\Delta y_{it} - \widehat{\rho} \Delta y_{it-1}}{\Delta x_{it}}$ to be contaminated. Independent AO contamination at rate ε yields a fraction of uncontaminated ratios equal to $(1-\varepsilon)^3$, hence breakdown requires $(1-\varepsilon)^3 \leq 0.5$, i.e. $\varepsilon \geq 1 - \sqrt[3]{0.5}$. This lower bound on the breakdown point is likely to be very crude since independent AO contamination induces a restricted class of contaminated distributions of $(\Delta y_{it} - \widehat{\rho} \Delta y_{it-1}, \Delta x_{it})$ while the breakdown point of the median-of-slopes in the iid case is 0.5 under arbitrary contamination of the joint distribution of the regressand and the regressor.

4.2 Ellyptically contoured errors

When the errors ε_{it} in (1) are non-normal, $\widehat{\rho}$ remains Fisher consistent for ρ if the joint distribution of $(\Delta y_{it}, \Delta y_{it-1})$ is elliptically contoured. This is the case when

$$\begin{pmatrix} y_{i0} \\ \varepsilon_i \end{pmatrix} \sim ECD\left(\mu_i, \Sigma_i, \phi\right),$$

$$\mu_i = \left(\frac{\alpha_i}{1-\rho}, 0, 0, ..., 0\right)', \qquad \Sigma_i = \sigma_i^2 \begin{pmatrix} (1-\rho^2)^{-1} & 0 \\ 0 & I_T \end{pmatrix},$$

[5] In the iid setting, the median-of-slopes has the minmax bias property (Martin, Yohai, and Zamar, 1989). Here, however, the setting is not iid.

where $\varepsilon_i = (\varepsilon_{i1}, ..., \varepsilon_{iT})'$ and ϕ is the characteristic generator because then

$$\begin{pmatrix} \Delta y_{it} \\ \Delta y_{it-1} \end{pmatrix} \sim ECD\left(0, \Omega_i, \phi\right), \qquad \Omega_i = \frac{\sigma_i^2}{1+\rho} \begin{pmatrix} 2 & \rho - 1 \\ \rho - 1 & 2 \end{pmatrix}.$$

Note that the scatter matrix Ω_i is the same as in (3). Defining $r = \frac{\rho - 1}{2}$ as before, we have

$$\begin{pmatrix} \Delta y_{it} - r \Delta y_{it-1} \\ \Delta y_{it-1} \end{pmatrix} \sim ECD\left(0, V_i, \phi_2\right), \qquad V_i = \frac{2\sigma_i^2}{1+\rho} \begin{pmatrix} 1 - r^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, the distribution of $(\Delta y_{it} - r \Delta y_{it-1}, \Delta y_{it-1})$ has equal orthant probabilities and, therefore,

$$\Pr\left(\frac{\Delta y_{it}}{\Delta y_{it-1}} \leq r\right) = \Pr\left(\frac{\Delta y_{it} - r \Delta y_{it-1}}{\Delta y_{it-1}} \leq 0\right) = \frac{1}{2}.$$

The Fisher consistency of $\widehat{\rho}$ follows. The robustness analysis parallels the analysis under normality but with the asymptotic bias calculations now being carried out using the cdf associated with ϕ instead of the standard normal cdf.

5 Simulations

We compared the efficiency and robustness of the median-based estimators, the two-step Blundell and Bond (1998) estimator (BB), and the ML estimator based on differenced data as suggested by Hsiao, Pesaran, and Tahmiscioglu (2002) (HPT). Below is a summary of simulation results, always with $N = 1000$ and $T = 5$, in settings with and without data contamination. We used 1000 Monte Carlo replications.[6]

[6] We refer to the cited papers for a description of the BB and HPT estimators. We also considered the robust generalized method of moments estimator proposed by Lucas, van Dijk, and Kloek (2007), which is a robust version of the Arellano and Bond (1991) estimator. However, the latter estimator is known to be severely biased when ρ is large and there is no contamination, a property that the former estimator inherits. We omit those simulation rsults.

AR(1). Table 1 gives results for the AR(1) model with uncontaminated data y_{it} generated according to (1)–(2) with $\sigma_i^2 = 1$ for all i and $\alpha_i \overset{\text{iid}}{\sim} N(0,1)$. Five percent contamination was introduced as $y_{it}^{.05} = y_{it} + a_{it}$ with mixed-normal AO: either iid draws $a_{it} \sim z_{it}N(0,100)$ with $z_{it} \sim$ Bernoulli(0.05) or patches $a_{it} \sim z_{it}N(0,100)$ of length $k = 3$ as described above with $z_{it} \sim$ Bernoulli$(1-\sqrt[3]{0.95})$.[7]

Table 1: simulations, AR(1) model

	$\widehat{\rho}$		BB		HPT	
ρ	mean	std	mean	std	mean	std
			no contamination			
.5	.50	.055	.50	.031	.50	.026
.9	.90	.056	.90	.048	.90	.027
			5% independent AO			
.5	.47	.054	.07	.022	.07	.033
.9	.82	.051	.10	.029	.12	.039
			5% patched AO			
.5	.57	.054	.76	.059	.79	.039
.9	.92	.046	.96	.032	.87	.038

The simulations confirm the asymptotic analysis. The bias of $\widehat{\rho}$ under independent AO contamination is always towards zero and is largest (in absolute value) when ρ is close to 1. The bias is always less than 0.15, which is approximately the fraction of contaminated ratios. In the case of patched outliers, $\widehat{\rho}$ has a mild positive bias. When there is no contamination, the BB and HPT estimators outperform $\widehat{\rho}$ in terms of efficiency. Unlike $\widehat{\rho}$, when a small fraction of large independent outliers is added to the data, the BB and HPT estimators are heavily biased towards 0. When a small fraction of patched outliers is added, $\widehat{\rho}$ and the BB and HPT estimators are nearly

[7] Absent contamination, the following invariances hold: (i) the median-based estimators are invariant with respect to all σ_i^2 and α_i; (ii) when $\sigma_i^2 = \sigma^2$, the HPT estimator is invariant with respect to all α_i and σ^2; (iii) when $\sigma_i^2 = \sigma^2$ and $\alpha_i \sim N(\mu_\alpha, \sigma_\alpha^2)$, the BB estimator depends on σ^2, μ_α, and σ_α^2 only through σ_α^2/σ^2. With contamination, all estimators are invariant under common scale transformations of α_i, ε_{it}, and a_{it}.

unbiased when ρ is close to 1; when $\rho = 0.5$, the biases are larger and $\widehat{\rho}$ has the least bias. Note that contamination affects the standard error of the BB and HPT estimators while hardly affecting that of $\widehat{\rho}$.

In other simulations we found that (i) in most cases with contamination varying N and T has little impact on the bias; (ii) $\widehat{\rho}$ and $\widehat{\rho}_{\cdot}$ have the same bias and standard error; (iii) $\widetilde{\rho}$ and $\widetilde{\rho}_{\cdot}$ have the same bias as $\widehat{\rho}$ but a smaller standard error (e.g. 8% less when $\rho = 0.5$; 30% less when $\rho = -0.5$).

AR(2). For the AR(2) model, we considered the BB and HPT estimators and the median-based estimators $\widehat{\rho}^a$ based on (9) and (10) and $\widehat{\rho}^b$ based on (9) and (11).[8] We generated uncontaminated AR(2) data and added 5% AO, using the same design as in the AR(1) case. On the whole, contamination makes the bias of all estimators more pronounced than in the AR(1) model. This is unsurprising because, intuitively, the estimators extract information from longer segments of data, which are more vulnerable to contamination. Table 2 gives results for two design points, $(\rho_1, \rho_2) = (0.7, -0.2)$ and $(\rho_1, \rho_2) = (-0.7, 0.2)$. When there is contamination, the median-based estimators are less biased than the BB and HPT estimators. The latter estimators are off the mark in several cases. Note also that, in the presence of patched outliers, all estimators have very different biases across the two design points. This does not occur under independent AO.

AR(1) with a covariate. Table 3 gives results for the AR(1) model with a covariate $x_{it} \overset{\text{iid}}{\sim} N(0, 1)$. We only considered independent AO that simultaneously affect y_{it} and x_{it} but are drawn independently from $N(0, 100)$. Other aspects of the design are as above. The median-based estimators $\widehat{\rho}$ and $\widehat{\beta}$ are defined by (13) (together with the reciprocal ratios) and (14). Concerning the estimation of ρ, the results are nearly the same as in the AR(1) model. In addition, they extend to the estimation of β. Under interpendent AO, the BB and HPT estimators are severely biased towards 0 while the median-based estimator is much less biased.

[8] The ratios defined in (9) and (10) were used together with their reciprocals. Note that $\widehat{\rho}^a$ and $\widehat{\rho}^b$ produce the same estimator of ρ_1.

Table 2: simulations, AR(2) model

ρ	$\widehat{\rho}^a$ mean	$\widehat{\rho}^a$ std	$\widehat{\rho}^b$ mean	$\widehat{\rho}^b$ std	BB mean	BB std	HPT mean	HPT std
				no contamination				
.7	.70	.050	.70	.050	.70	.039	.70	.025
−.2	−.20	.058	−.20	.045	−.20	.027	−.20	.025
−.7	−.70	.057	−.70	.057	−.70	.037	−.70	.032
.2	.20	.059	.20	.059	.20	.037	.20	.033
				5% independent AO				
.7	.50	.054	.50	.054	.15	.037	.14	.056
−.2	−.32	.058	−.09	.038	.03	.028	.03	.049
−.7	−.48	.055	−.48	.055	−.23	.054	−.20	.044
.2	.40	.056	.37	.055	.43	.060	.37	.056
				5% patched AO				
.7	.71	.051	.71	.051	.76	.096	.82	.032
−.2	−.21	.057	−.17	.043	−.14	.042	−.16	.038
−.7	−.53	.058	−.53	.058	.11	.138	.29	.069
.2	.34	.059	.35	.058	.90	.125	.88	.063

Note: $\widehat{\rho}^a$ is based on (9), (10); $\widehat{\rho}^b$ is based on (9), (11).

Table 3: simulations, AR(1) model with covariate

$\binom{\rho}{\beta}$	$\widehat{\rho}, \widehat{\beta}$ mean	$\widehat{\rho}, \widehat{\beta}$ std	BB mean	BB std	HPT mean	HPT std
			no contamination			
.5	.50	.048	.50	.020	.50	.019
1	1.00	.032	1.00	.024	1.00	.020
.9	.90	.054	.91	.023	.90	.025
1	1.00	.033	1.00	.024	1.00	.021
			5% independent AO			
.5	.47	.048	.12	.026	.13	.031
1	.91	.034	.21	.082	.15	.064
.9	.81	.049	.17	.044	.21	.041
1	.89	.034	.18	.093	.11	.065

6 Conclusion

We proposed an estimator based on sample medians of ratios for outlier-robust estimation of linear dynamic fixed effects panel models. The case of estimating the AR(1) coefficient was studied in detail, and the estimator was shown to have attractive robustness properties under additive outlier contamination.

The initial observations were assumed to be drawn from the stationary distributions when $\rho < 1$, that is, the start-up of the processes has to lie in the distant past. In applications, it will often be possible to judge the validity of this assumption. Moreover, the assumption has the testable implication that the time series of the cross-sectional locations and scales of Δy_{it} are zero and constant, respectively. Hence the time series of cross-sectional medians and median absolute deviations of Δy_{it} should be close to zero and nearly constant, respectively. If this is called into doubt, one may discard the earliest observations (i.e. those with small t). The bias of the median-based estimator arising from non-stationary initial observations vanishes at an exponential rate as the number of discarded observations increases.

Appendix

Proof of Lemma 1. We prove the result for $\widehat{r}$ in two steps. First, we show that

$$a_N\left(\widehat{r}-r\right)\stackrel{d}{\to}N\left(0,V_T\right)$$

where

$$a_N=\frac{2}{\sqrt{N\left(T-2\right)}}\sum_{i=1}^{N}\sum_{t=3}^{T}f_i\left(0\right)\left|\Delta y_{it-1}\right|$$

and f_i is the density function of $u_{it}=\Delta y_{it}-r\Delta y_{it-1}$. In the second step, we conclude the proof by showing that, as $N\to\infty$,

$$\frac{a_N}{\sqrt{N\left(T-2\right)}}\stackrel{p}{\to}2f_i\left(0\right)E\left|\Delta y_{it}\right|=\sqrt{\frac{4}{\pi^2(1-r^2)}}. \tag{15}$$

Let $r_{it} = \Delta y_{it}/\Delta y_{it-1}$ and let I be the indicator function. Then

$$\begin{aligned}\Pr[a_N(\widehat{r}-r) \le z] &= \Pr[\widehat{r} \le r + a_N^{-1} z] \\ &= \Pr\left[\frac{1}{N(T-2)}\sum_{i=1}^{N}\sum_{t=3}^{T} I\left(r_{it} \le r + a_N^{-1}z\right) \ge \frac{1}{2}\right].\end{aligned}$$

Following So and Shin (2001),

$$I\left(r_{it} \le r + a_N^{-1}z\right) = I\left(\operatorname{sign}\left(u_{it} - a_N^{-1} z \Delta y_{it-1}\right)\operatorname{sign}\left(\Delta y_{it-1}\right) \le 0\right)$$

and

$$\Pr[a_N(\widehat{r}-r) \le z] = \Pr[A + R \le 0],$$

where

$$\begin{aligned}A &= \frac{1}{\sqrt{N(T-2)}}\sum_{i=1}^{N}\sum_{t=3}^{T}\operatorname{sign}(u_{it})\operatorname{sign}(\Delta y_{it-1}), \\ R &= \frac{1}{\sqrt{N(T-2)}}\sum_{i=1}^{N}\sum_{t=3}^{T}\left[\operatorname{sign}\left(u_{it} - a_N^{-1}z\Delta y_{it-1}\right) - \operatorname{sign}(u_{it})\right]\operatorname{sign}(\Delta y_{it-1}).\end{aligned}$$

As $N \to \infty$, $A \to_d N(0, V_T)$ and $a_N^{-1} \to_p 0$. Set $R = M + \Delta$, where

$$\begin{aligned}M &= \frac{1}{\sqrt{N(T-2)}}\sum_{i=1}^{N}\sum_{t=3}^{T}\left\{E\left[\operatorname{sign}\left(u_{it} - a_N^{-1}z\Delta y_{it-1}\right) - \operatorname{sign}(u_{it})\,|\Delta y_{it-1}\right]\right. \\ &\qquad \left.\times \operatorname{sign}(\Delta y_{it-1})\right\} \\ &= \frac{-2}{\sqrt{N(T-2)}}\sum_{i=1}^{N}\sum_{t=3}^{T}\left[F_i\left(a_N^{-1}z\Delta y_{it-1}\right) - F_i(0)\right]\operatorname{sign}(\Delta y_{it-1}) \\ &= \frac{-2}{\sqrt{N(T-2)}}\sum_{i=1}^{N}\sum_{t=3}^{T} f_i(0)\, a_N^{-1} z\,|\Delta y_{it-1}| + o_p(1) \\ &= -z + o_p(1)\end{aligned}$$

and F_i is the distribution function of $u_{it} = \Delta y_{it} - r\Delta y_{it-1}$. For Δ, we have $E\Delta^2 = (N(T-2))^{-1}\sum_{i=1}^{N} E\left(\sum_{t=3}^{T}\Delta_{it}\right)^2$, where

$$\begin{aligned}\Delta_{it} &= \left[\operatorname{sign}\left(u_{it} - a_N^{-1}z\Delta y_{it-1}\right) - \operatorname{sign}(u_{it})\right]\operatorname{sign}(\Delta y_{it-1}) \\ &\quad - E\left[\operatorname{sign}\left(u_{it} - a_N^{-1}z\Delta y_{it-1}\right) - \operatorname{sign}(u_{it})\,|\Delta y_{it-1}\right]\operatorname{sign}(\Delta y_{it-1}).\end{aligned}$$

As $N \to \infty$, $\Delta_{it} \xrightarrow{p} 0$, $E\Delta^2 \to 0$, $\Delta = o_p(1)$, and $A + R = A - z + o_p(1)$. As a result, $a_N(\hat{r} - r) \to_d N(0, V_T)$. To prove (15), recall (3), from which we obtain

$$\Delta y_{it} \sim N\left(0, \frac{\sigma_i^2}{1+r}\right), \qquad u_{it} \sim N(0, (1-r)\sigma_i^2),$$

and

$$E|\Delta y_{it}| = \sqrt{\frac{2\sigma_i^2}{\pi(1+r)}}, \qquad f_i(0) = \sqrt{\frac{1}{2\pi(1-r)\sigma_i^2}}.$$

Clearly, $f_i(0)|\Delta y_{it-1}|$ is identically distributed for all i and t, and has mean $f_i(0)E|\Delta y_{it}| = (\pi^2(1-r^2))^{-1/2}$. Now (15) follows, which completes the proof for $\widehat{r}$. The proof for $\widehat{r}_Z$ is identical. ■

Proof of Lemma 2. By a standard argument, if $N \to \infty$, then for any $t, t' \geq 3$,

$$\sqrt{N}\begin{pmatrix} \widehat{r}_t - r \\ \widehat{r}_{t'} - r \end{pmatrix} = \frac{1}{2g(r)\sqrt{N}} \sum_{i=1}^{N} \begin{pmatrix} \text{sign}\left(\Delta y_{it}/\Delta y_{it-1} - r\right) \\ \text{sign}\left(\Delta y_{it'}/\Delta y_{it'-1} - r\right) \end{pmatrix} + o_p(1),$$

where $g(r) = \pi^{-1}(1-r^2)^{-1/2}$, the density of $\Delta y_{it}/\Delta y_{it-1}$ at r. Recalling $\text{sign}\left(\Delta y_{it}/\Delta y_{it-1} - r\right) = \text{sign}(u_{it})\,\text{sign}(\Delta y_{it-1})$, we find

$$\sqrt{N(T-2)}\,(\widehat{r}_{\cdot} - r) = \frac{\pi(1-r^2)^{1/2}}{2\sqrt{N(T-2)}} \sum_{i=1}^{N}\sum_{t=3}^{T} \text{sign}(u_{it})\,\text{sign}(\Delta y_{it-1}) + o_p(1), \tag{16}$$

and the result follows for fixed T and $N \to \infty$. The proof is complete by noting that (16) also holds when $T \to \infty$ and N is fixed or $N \to \infty$. ■

Bias of $\widehat{\rho}$ under independent AO with point-mass distribution. To solve (6) for b, we need to evaluate

$$\begin{aligned} \Pr\left[\frac{\Delta y_{it}^{\zeta,\varepsilon} - r\Delta y_{it-1}^{\zeta,\varepsilon}}{\Delta y_{it-1}^{\zeta,\varepsilon}} \leq b\right] &= \Pr\left[\frac{u_{it} + \Delta a_{it} - r\Delta a_{it-1}}{\Delta y_{it-1} + \Delta a_{it-1}} \leq b\right] \\ &= ((1-\varepsilon)^3 + \varepsilon^3)A + (1-\varepsilon)^2\varepsilon B + (1-\varepsilon)\varepsilon^2 C, \end{aligned}$$

where $u_{it} = \Delta y_{it} - r\Delta y_{it-1}$ and

$$\begin{aligned}
A &= \Pr\left[\frac{u_{it}}{\Delta y_{it-1}} \le b\right], \\
B &= \Pr\left[\frac{u_{it}+\zeta}{\Delta y_{it-1}} \le b\right] + \Pr\left[\frac{u_{it}-(1+r)\zeta}{\Delta y_{it-1}+\zeta} \le b\right] + \Pr\left[\frac{u_{it}+r\zeta}{\Delta y_{it-1}-\zeta} \le b\right], \\
C &= \Pr\left[\frac{u_{it}-r\zeta}{\Delta y_{it-1}+\zeta} \le b\right] + \Pr\left[\frac{u_{it}+(1+r)\zeta}{\Delta y_{it-1}-\zeta} \le b\right] + \Pr\left[\frac{u_{it}-\zeta}{\Delta y_{it-1}} \le b\right].
\end{aligned}$$

Thus, we need to compute probabilities of the form

$$F(k,l,b) = \Pr\left[\frac{u_{it}+k}{\Delta y_{it-1}-l} \le b\right].$$

for given k, l, and b. Using $\sigma_i^2 = \sigma^2$ and, from (3),

$$\begin{pmatrix} u_{it} \\ \Delta y_{it-1} \end{pmatrix} \sim N\left(0, \frac{\sigma^2}{1+r}\begin{pmatrix} 1-r^2 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

we have, on standardizing u_{it} and Δy_{it-1},

$$F(k,l,b) = \Pr\left[\frac{X+k'}{Z-l'} \le b'\right], \tag{17}$$

where X and Z are independent $N(0,1)$ variates, and

$$k' = \frac{k}{\sigma\sqrt{1-r}}, \qquad l' = \frac{l\sqrt{1+r}}{\sigma}, \qquad b' = \frac{b}{\sqrt{1-r^2}}.$$

Hence,

$$\begin{aligned}
F(k,l,b) &= \Pr\left[X+k' \le b'(Z-l'), Z-l' > 0\right] \\
&\quad + \Pr\left[X+k' \ge b'(Z-l'), Z-l' < 0\right] \\
&= \int_{l'}^{\infty} \Phi(b'z - b'l' - k')\phi(z)dz + \int_{-\infty}^{l'} \Phi(-b'z + b'l' + k')\phi(z)dz,
\end{aligned}$$

where where Φ and ϕ are the standard normal cdf and pdf. The required probabilities $F(k,l,b)$ can be computed numerically, and (6) solved for b. Note that

$$A = \Pr\left[\frac{X}{Z} \le \frac{b}{\sqrt{1-r^2}}\right] = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{b}{\sqrt{1-r^2}}. \tag{18}$$

Further, by (17) and an argument of symmetry, $F(k,l,b) = F(-k,-l,b)$. Thus, $B = C$ and (6) reduces to

$$A(r,b) + \varepsilon(1-\varepsilon)(B(r,\zeta,b) - 3A(r,b)) = \frac{1}{2}, \tag{19}$$

where the arguments of A and B indicate functional dependencies, and $B(r,\zeta,b) = B(r,-\zeta,b)$. Then $\text{Bias}(\widehat{\rho};\rho,\zeta,\varepsilon) = 2b(r,\zeta,\varepsilon)$ where $b(r,\zeta,\varepsilon)$ uniquely solves (19). $\text{MB}(\widehat{\rho};\rho,\varepsilon)$ follows on maximizing $|b(r,\zeta,\varepsilon)|$ numerically with respect to ζ. ■

Proof of Theorem 1. From (19), $b(r,\zeta,\varepsilon) = b(r,\zeta,1-\varepsilon) = b(r,-\zeta,\varepsilon)$ and (i) follows. Because $b(r,\zeta,\varepsilon)$ depends on ζ and σ only through $|\zeta/\sigma|$, without loss of generality we set $\zeta > 0$ and $\sigma = 1$ in the remainder of the proof. From (19), we see that, at the solution, $A(r,b) - \frac{1}{2}$ and $B(r,\zeta,b) - 3A(r,b)$ have opposite signs, and similarly for $A(r,b) - \frac{1}{2}$ and $B(r,\zeta,b) - \frac{3}{2}$. Both $A(r,b)$ and $B(r,\zeta,b)$ are increasing in b, so either $|b(r,\zeta,\varepsilon)|$ is increasing in $\varepsilon(1-\varepsilon)$ or $b(r,\zeta,\varepsilon) = 0$ for all ε. Because $A(r,0) = \frac{1}{2}$, $B(r,0,0) = \frac{3}{2}$, and the left-hand side of (19) is increasing in b, to prove (ii) and (iii) we only need to show that

$$B(r,\zeta,0) \gtreqless \frac{3}{2} \text{ if } r \gtreqless -\frac{1}{2}. \tag{20}$$

We can write $B(r,\zeta,0)$ as

$$\begin{aligned} B(r,\zeta,0) &= \Pr\left[\frac{X + \frac{\zeta}{\sqrt{1-r}}}{Z} \le 0\right] + \Pr\left[\frac{X + \frac{(1+r)\zeta}{\sqrt{1-r}}}{Z - \sqrt{1+r\zeta}} \le 0\right] \\ &\quad + \Pr\left[\frac{X + \frac{r\zeta}{\sqrt{1-r}}}{Z - \sqrt{1+r\zeta}} \le 0\right], \end{aligned} \tag{21}$$

where X and Z are independent $N(0,1)$ variates. By (27)–(28) of Lemma 4 below, $B(r,\zeta,0) \gtreqless \frac{3}{2}$ if $\frac{(1+r)\zeta}{\sqrt{1-r}} + \frac{r\zeta}{\sqrt{1-r}} \gtreqless 0$. Hence (20) follows, completing the proof of (ii) and (iii). Given (iii), (iv) is equivalent to

$$r + b(r,\zeta,\varepsilon) \gtreqless -\frac{1}{2} \text{ if } r \gtreqless -\frac{1}{2}.$$

In turn, because $A(r,b)$ and $B(r,\zeta,b)$ are increasing in b, this is equivalent to

$$A(r,-\tfrac{1}{2}-r)+\varepsilon(1-\varepsilon)(B(r,\zeta,-\tfrac{1}{2}-r)-3A(r,-\tfrac{1}{2}-r)) \lesseqqgtr \frac{1}{2} \text{ if } r \gtreqqless -\frac{1}{2}, \quad (22)$$

Clearly,

$$A(r,-\tfrac{1}{2}-r) = \Pr\left[\frac{X}{Z} \leq \frac{-\frac{1}{2}-r}{\sqrt{1-r^2}}\right] \lesseqqgtr \frac{1}{2} \text{ if } r \gtreqqless -\frac{1}{2}, \quad (23)$$

because $-\frac{1}{2}-r \lesseqqgtr 0$ if $r \gtreqqless -\frac{1}{2}$. Similarly, by (27) of Lemma 4, the first term of $B(r,\zeta,-\frac{1}{2}-r)$ is

$$\Pr\left[\frac{X+\frac{\zeta}{\sqrt{1-r}}}{Z} \leq \frac{-\frac{1}{2}-r}{\sqrt{1-r^2}}\right] \lesseqqgtr \frac{1}{2} \text{ if } r \gtreqqless -\frac{1}{2}.$$

The sum of the last two terms of $B(r,\zeta,-\frac{1}{2}-r)$ is

$$\Pr\left[\frac{X+\frac{(1+r)\zeta}{\sqrt{1-r}}}{Z-\sqrt{1+r}\zeta} \leq \frac{-\frac{1}{2}-r}{\sqrt{1-r^2}}\right]+\Pr\left[\frac{X+\frac{r\zeta}{\sqrt{1-r}}}{Z-\sqrt{1+r}\zeta} \leq \frac{-\frac{1}{2}-r}{\sqrt{1-r^2}}\right] \lesseqqgtr 1 \text{ if } r \gtreqqless -\frac{1}{2}$$

by (29) of Lemma 4 with $a=\sqrt{1+r}\zeta$, $c=\frac{1+r}{\sqrt{1-r^2}}$, and $c'=\frac{r}{\sqrt{1-r^2}}$. Hence,

$$B(r,\zeta,-\tfrac{1}{2}-r) \lesseqqgtr \frac{3}{2} \text{ if } r \gtreqqless -\frac{1}{2}, \quad (24)$$

which, combined with (23), gives (22), thus completing the proof of (iv). To prove (v), let $0<\rho<1$. Then $-\frac{1}{2}<r<0$ and $-\frac{1}{2}<b(r,\zeta,\varepsilon)<0$. We need to show that $b_r<0$ (here and later, subscripts denote partial derivatives). On differentiating (19), we obtain

$$b_r = -\frac{(1-3\varepsilon(1-\varepsilon))\,A_r+\varepsilon(1-\varepsilon)B_r}{(1-3\varepsilon(1-\varepsilon))\,A_b+\varepsilon(1-\varepsilon)B_b}.$$

Since the denominator of b_r is always positive, the proof is complete if we show that $A_r>0$ and $B_r>0$. By (18), $A_r>0$ because $b/\sqrt{1-r^2}$ is increasing in r and $\arctan(\cdot)$ is increasing. With regard to B_r, write B as

$$\begin{aligned} B &= \Pr\left[\frac{X+\frac{\zeta}{\sqrt{1-r}}}{Z} \le \frac{b}{\sqrt{1-r^2}}\right] + \Pr\left[\frac{X+\frac{(1+r)\zeta}{\sqrt{1-r}}}{Z-\sqrt{1+r}\zeta} \le \frac{b}{\sqrt{1-r^2}}\right] \\ &\quad + \Pr\left[\frac{X+\frac{r\zeta}{\sqrt{1-r}}}{Z-\sqrt{1+r}\zeta} \le \frac{b}{\sqrt{1-r^2}}\right] \\ &= B^{(1)} + B^{(2)} + B^{(3)} \qquad \text{(say)}, \end{aligned}$$

where X and Z are independent $N(0,1)$ variates. We will show that $B_r^{(1)} > 0$ and $B_r^{(2)} + B_r^{(3)} > 0$ by repeatedly using the property that $\phi(\beta z - \alpha)\phi(z) = c\phi\left(\frac{z-\mu}{\omega}\right)$ with $\mu = \frac{\alpha\beta}{1+\beta^2}$, $\omega^2 = \frac{1}{1+\beta^2}$, and $c = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}\alpha^2\omega^2) > 0$. Let

$$a = \frac{b}{\sqrt{1-r^2}}z - \frac{1}{\sqrt{1-r}}\zeta, \qquad a_r = \frac{br}{(1-r^2)^{3/2}}z - \frac{1}{2(1-r)^{3/2}}\zeta.$$

Then

$$\begin{aligned} B^{(1)} &= \int_0^\infty \Phi(a)\phi(z)dz + \int_{-\infty}^0 \Phi(-a)\phi(z)dz, \\ B_r^{(1)} &= \frac{br}{(1-r^2)^{3/2}}\left[\int_0^\infty \phi(a)\phi(z)zdz - \int_{-\infty}^0 \phi(a)\phi(z)zdz\right] \\ &\quad - \frac{\zeta}{2(1-r)^{3/2}}\left[\int_0^\infty \phi(a)\phi(z)dz - \int_{-\infty}^0 \phi(a)\phi(z)dz\right]. \end{aligned}$$

Here we can write $\phi(a)\,\phi(z)$ as $c\phi\left(\frac{z-\mu}{\omega}\right)$ with $\mu < 0$ and $c, \omega > 0$, to see that the two bracketed expressions are, respectively,

$$\begin{aligned} 2\int_0^\infty c\phi\left(\frac{z-\mu}{\omega}\right)zdz - \int_{-\infty}^\infty c\phi\left(\frac{z-\mu}{\omega}\right)zdz &> 0, \\ \int_0^\infty c\phi\left(\frac{z-\mu}{\omega}\right)dz - \int_{-\infty}^0 c\phi\left(\frac{z-\mu}{\omega}\right)dz &< 0. \end{aligned}$$

Hence $B_r^{(1)} > 0$. On redefining a as

$$a = \frac{b}{\sqrt{1-r^2}}z - \frac{1+r+b}{\sqrt{1-r}}\zeta, \qquad a_r = \frac{br}{(1-r^2)^{3/2}}z - \frac{3+b-r}{2(1-r)^{3/2}}\zeta,$$

we have

$$
\begin{aligned}
B^{(2)} &= \int_{\sqrt{1+r}\zeta}^{\infty} \Phi(a)\,\phi(z)\,dz + \int_{-\infty}^{\sqrt{1+r}\zeta} \Phi(-a)\,\phi(z)\,dz, \\
B_r^{(2)} &= \frac{\zeta\phi\left(\sqrt{1+r}\zeta\right)}{2\sqrt{1+r}} \left(\Phi\left(\frac{1+r}{\sqrt{1-r}}\zeta\right) - \Phi\left(-\frac{1+r}{\sqrt{1-r}}\zeta\right)\right) \\
&\quad + \frac{br}{\left(1-r^2\right)^{3/2}} \left[\int_{\sqrt{1+r}\zeta}^{\infty} \phi(a)\,\phi(z)\,zdz - \int_{-\infty}^{\sqrt{1+r}\zeta} \phi(a)\,\phi(z)\,zdz\right] \\
&\quad - \frac{(3+b-r)\,\zeta}{2\,(1-r)^{3/2}} \left[\int_{\sqrt{1+r}\zeta}^{\infty} \phi(a)\,\phi(z)\,dz - \int_{-\infty}^{\sqrt{1+r}\zeta} \phi(a)\,\phi(z)\,dz\right].
\end{aligned}
$$

Again, $\phi(a)\,\phi(z)$ is $c\phi\left(\frac{z-\mu}{\omega}\right)$ with $\mu < 0$ and $c, \omega > 0$ and the bracketed expressions can be signed as before, resulting in

$$
B_r^{(2)} > \frac{\zeta\phi\left(\sqrt{1+r}\zeta\right)}{2\sqrt{1+r}} \left(\Phi\left(\frac{1+r}{\sqrt{1-r}}\zeta\right) - \Phi\left(-\frac{1+r}{\sqrt{1-r}}\zeta\right)\right). \tag{25}
$$

On redefining a again, now as

$$
a = \frac{b}{\sqrt{1-r^2}}z - \frac{r+b}{\sqrt{1-r}}\zeta, \qquad a_r = \frac{br}{\left(1-r^2\right)^{3/2}}z - \frac{2+b-r}{2\,(1-r)^{3/2}}\zeta,
$$

we have

$$
\begin{aligned}
B^{(3)} &= \int_{\sqrt{1+r}\zeta}^{\infty} \Phi(a)\,\phi(z)\,dz + \int_{-\infty}^{\sqrt{1+r}\zeta} \Phi(-a)\,\phi(z)\,dz, \\
B_r^{(3)} &= \frac{\zeta\phi\left(\sqrt{1+r}\zeta\right)}{2\sqrt{1+r}} \left(\Phi\left(\frac{r}{\sqrt{1-r}}\zeta\right) - \Phi\left(-\frac{r}{\sqrt{1-r}}\zeta\right)\right) \\
&\quad + \frac{br}{\left(1-r^2\right)^{3/2}} \left[\int_{\sqrt{1+r}\zeta}^{\infty} \phi(a)\,\phi(z)\,zdz - \int_{-\infty}^{\sqrt{1+r}\zeta} \phi(a)\,\phi(z)\,zdz\right] \\
&\quad - \frac{(2+b-r)\,\zeta}{2\,(1-r)^{3/2}} \left[\int_{\sqrt{1+r}\zeta}^{\infty} \phi(a)\,\phi(z)\,dz - \int_{-\infty}^{\sqrt{1+r}\zeta} \phi(a)\,\phi(z)\,dz\right].
\end{aligned}
$$

Here, $\phi(a)\,\phi(z)$ is $c\phi\left(\frac{z-\mu}{\omega}\right)$ with $\mu = \frac{b(r+b)}{1+b^2-r^2}\sqrt{1+r}\zeta > 0$ and $c, \omega > 0$.

Write $B_r^{(3)} = T^{(1)} + T^{(2)} + T^{(3)}$ (say). The bracketed expression in $T^{(2)}$ is

$$\int_{\sqrt{1+r}\zeta}^{\infty} c\phi\left(\frac{z-\mu}{\omega}\right) zdz - \int_{-\infty}^{\sqrt{1+r}\zeta} c\phi\left(\frac{z-\mu}{\omega}\right) zdz$$

$$> \ -\int_{0}^{\sqrt{1+r}\zeta} c\phi\left(\frac{z-\mu}{\omega}\right) zdz > -\zeta\int_{0}^{\sqrt{1+r}\zeta} c\phi\left(\frac{z-\mu}{\omega}\right) dz.$$

Hence, using $\frac{2+b-r}{2(1-r)^{3/2}} > \frac{br}{(1-r^2)^{3/2}} > 0$,

$$\begin{aligned} T^{(2)} &> -\frac{(2+b-r)\,\zeta}{2\,(1-r)^{3/2}}\int_{0}^{\sqrt{1+r}\zeta} c\phi\left(\frac{z-\mu}{\omega}\right) dz, \\ T^{(2)} + T^{(3)} &> \frac{(2+b-r)\,\zeta}{2\,(1-r)^{3/2}}\left[\int_{-\infty}^{\sqrt{1+r}\zeta} c\phi\left(\frac{z-\mu}{\omega}\right) dz - \int_{0}^{\infty} c\phi\left(\frac{z-\mu}{\omega}\right) dz\right]. \end{aligned}$$

Here, the expression in brackets can be written as

$$\int_{\mu}^{\sqrt{1+r}\zeta} c\phi\left(\frac{z-\mu}{\omega}\right) dz - \int_{0}^{\mu} c\phi\left(\frac{z-\mu}{\omega}\right) dz = \int_{2\mu}^{\sqrt{1+r}\zeta} c\phi\left(\frac{z-\mu}{\omega}\right) dz > 0,$$

because $2\mu < \sqrt{1+r}\zeta$. Hence

$$B_r^{(3)} > \frac{\zeta\phi\left(\sqrt{1+r}\zeta\right)}{2\sqrt{1+r}}\left(\Phi\left(\frac{r}{\sqrt{1-r}}\zeta\right) - \Phi\left(-\frac{r}{\sqrt{1-r}}\zeta\right)\right). \tag{26}$$

Combining (25)–(26) with

$$\Phi\left(\frac{1+r}{\sqrt{1-r}}\zeta\right) > \Phi\left(-\frac{r}{\sqrt{1-r}}\zeta\right), \qquad \Phi\left(\frac{r}{\sqrt{1-r}}\zeta\right) > \Phi\left(-\frac{1+r}{\sqrt{1-r}}\zeta\right),$$

it follows that $B_r^{(2)} + B_r^{(3)} > 0$, completing the proof of Theorem 1. ■

Lemma 3 *Let X and Z be independent $N(0,1)$ variates. Let c, c', and a be constants, with $a > 0$. Then*

$$\Pr\left[\frac{X+c}{Z} \le 0\right] = \frac{1}{2}, \tag{27}$$

$$\Pr\left[\frac{X+c}{Z-a} \le 0\right] + \Pr\left[\frac{X+c'}{Z-a} \le 0\right] \gtreqless 1 \text{ if } c + c' \gtreqless 0, \tag{28}$$

$$\Pr\left[\frac{X+ac}{Z-a} \le -\frac{c+c'}{2}\right] + \Pr\left[\frac{X+ac'}{Z-a} \le -\frac{c+c'}{2}\right] \lesseqgtr 1 \text{ if } c + c' \gtreqless 0. \tag{29}$$

Proof. We have

$$\Pr\left[\frac{X+c}{Z}\le 0\right]=\Phi(-c)\Phi(0)+\Phi(c)\Phi(0)=\frac{1}{2}.$$

Further,

$$\begin{aligned}\Pr\left[\frac{X+c}{Z-a}\le 0\right] &= \Phi(-c)\Phi(-a)+\Phi(c)\Phi(a),\\ \Pr\left[\frac{X+c'}{Z-a}\le 0\right] &= \Phi(-c')\Phi(-a)+\Phi(c')\Phi(a).\end{aligned}$$

Simple algebra gives

$$\Pr\left[\frac{X+c}{Z-a}\le 0\right]+\Pr\left[\frac{X+c'}{Z-a}\le 0\right]=1+\left(\Phi(c)-\Phi(-c')\right)\left(\Phi(a)-\Phi(-a)\right),$$

from which (28) follows. Next,

$$\begin{aligned}\Pr\left[\frac{X+ac}{Z-a}\le -\frac{c+c'}{2}\right] &= \int_a^\infty \Phi\left(-\frac{c+c'}{2}z-\frac{c-c'}{2}a\right)\phi(z)dz\\ &\quad+\int_{-\infty}^a \Phi\left(\frac{c+c'}{2}z+\frac{c-c'}{2}a\right)\phi(z)dz,\\ \Pr\left[\frac{X+ac'}{Z-a}\le -\frac{c+c'}{2}\right] &= \int_a^\infty \Phi\left(-\frac{c+c'}{2}z+\frac{c-c'}{2}a\right)\phi(z)dz\\ &\quad+\int_{-\infty}^a \Phi\left(\frac{c+c'}{2}z-\frac{c-c'}{2}a\right)\phi(z)dz.\end{aligned}$$

Hence, letting $\xi=\frac{c+c'}{2}$ and $\delta=\frac{c-c'}{2}a$,

$$\begin{aligned}&\Pr\left[\frac{X+ac}{Z-a}\le -\xi\right]+\Pr\left[\frac{X+ac'}{Z-a}\le -\xi\right]\\ &= 1-\int_a^\infty \Phi\left(\xi z+\delta\right)\phi(z)dz+\int_{-\infty}^a \Phi\left(\xi z+\delta\right)\phi(z)dz\\ &\quad+\int_a^\infty \Phi\left(-\xi z+\delta\right)\phi(z)dz-\int_{-\infty}^a \Phi\left(-\xi z+\delta\right)\phi(z)dz\\ &= 1-\int_a^\infty \Phi\left(\xi z+\delta\right)\phi(z)dz+\int_{-\infty}^{-a} \Phi\left(\xi z+\delta\right)\phi(z)dz\\ &\quad+\int_a^\infty \Phi\left(-\xi z+\delta\right)\phi(z)dz-\int_{-\infty}^{-a} \Phi\left(-\xi z+\delta\right)\phi(z)dz\\ &\lesseqqgtr 1 \text{ if } \xi \gtreqqless 0,\end{aligned}$$

and (29) follows. ■

Proof of Theorem 2. The influence function is

$$\mathrm{IF}(\widehat{\rho};\rho,\zeta) = 2\,\frac{\partial b(r,\zeta,\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0}$$

where $b(r,\zeta,\varepsilon)$ solves (19). Differentiate (19) and evaluate the result at $\varepsilon = 0$ to obtain

$$\frac{\partial b(r,\zeta,\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0} = \frac{3A(r,0) - B(r,\zeta,0)}{A_b(r,0)}.$$

Recall $A(r,0) = \frac{1}{2}$. From (18), $A_b(r,0) = \pi^{-1}(1-r^2)^{-1/2}$. Now use (21), with ζ replaced with ζ/σ on its right-hand side, to write $B(r,\zeta,0)$ along the lines of the proof of Lemma 4 as

$$\begin{aligned} B(r,\zeta,0) &= \frac{3}{2} + \left[\Phi\left(\frac{(1+r)\zeta}{\sigma\sqrt{1-r}}\right) - \Phi\left(\frac{-r\zeta}{\sigma\sqrt{1-r}}\right)\right] \\ &\qquad \times \left[\Phi\left(\frac{\sqrt{1+r}\zeta}{\sigma}\right) - \Phi\left(\frac{-\sqrt{1+r}\zeta}{\sigma}\right)\right]. \end{aligned}$$

The result follows. ■

Bias of $\widehat{\rho}$ under independent AO with arbitrary distribution. We need to solve

$$\begin{aligned} \Pr\left[\frac{\Delta y_{it}^{\zeta,\varepsilon} - r\Delta y_{it-1}^{\zeta,\varepsilon}}{\Delta y_{it-1}^{\zeta,\varepsilon}} \le b\right] &= (1-\varepsilon)^3 A + (1-\varepsilon)^2\varepsilon B + (1-\varepsilon)\varepsilon^2 C + \varepsilon^3 D \\ &= \frac{1}{2} \qquad (30) \end{aligned}$$

for b. Here, A is as before,

$$\begin{aligned} B &= \Pr\left[\frac{u_{it}+\zeta_1}{\Delta y_{it-1}} \le b\right] + \Pr\left[\frac{u_{it}-(1+r)\zeta_1}{\Delta y_{it-1}+\zeta_1} \le b\right] + \Pr\left[\frac{u_{it}+r\zeta_1}{\Delta y_{it-1}-\zeta_1} \le b\right], \\ C &= \Pr\left[\frac{u_{it}+\zeta_1-(1+r)\zeta_2}{\Delta y_{it-1}+\zeta_2} \le b\right] + \Pr\left[\frac{u_{it}+\zeta_1+r\zeta_2}{\Delta y_{it-1}-\zeta_2} \le b\right] \\ &\quad + \Pr\left[\frac{u_{it}-(1+r)\,\zeta_1+r\zeta_2}{\Delta y_{it-1}+\zeta_1-\zeta_2} \le b\right], \\ D &= \Pr\left[\frac{u_{it}+z_1-(1+r)\,\zeta_2+r\zeta_3}{\Delta y_{it-1}+\zeta_2-\zeta_3} \le b\right], \end{aligned}$$

and ζ_1, ζ_2, ζ_3 are iid with cdf G_ζ. To solve (30), we need to evaluate A through D. A is computed using (18). To compute B, C, and D, we need to compute probabilities of the form

$$\Pr[K \le b], \qquad K = \frac{u_{it} + k_1\zeta_1 + k_2\zeta_2 + k_3\zeta_3}{\Delta y_{it-1} - l_1\zeta_2 - l_2\zeta_3},$$

for given constants k_j and l_j. Using the methods described above, it is easy to compute

$$\Pr[K \le b|\zeta_1 = z_1, \zeta_2 = z_2, \zeta_3 = z_3] = F(k_1z_1 + k_2z_2 + k_3z_3, l_1z_2 + l_2z_3, b).$$

Then, by the independence of $\zeta_1, \zeta_2, \zeta_3$,

$$\Pr[K \le b] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pr[K \le b|\zeta_1 = z_1, \zeta_2 = z_2, \zeta_3 = z_3]\, dG_\zeta(z_1)\, dG_\zeta(z_2)\, dG_\zeta(z_3),$$

which can be computed numerically. ■

Proof of Theorem 3. As before,

$$\begin{aligned} A(r,0) &= \frac{1}{2}, \\ A(r, -\tfrac{1}{2} - r) &\lesseqqgtr \frac{1}{2} \text{ if } r \gtreqqless -\frac{1}{2}, \end{aligned}$$

and, in view of (20) and (24),

$$\begin{aligned} B(r, G_\zeta, 0) &\gtreqqless \frac{3}{2} \text{ if } r \gtreqqless -\frac{1}{2}, \\ B\left(r, G_\zeta, -\tfrac{1}{2} - r\right) &\lesseqqgtr \frac{3}{2} \text{ if } r \gtreqqless -\frac{1}{2}. \end{aligned}$$

Write $C(r, G_\zeta, b) = \sum_{j=1}^{3} C_j(r, G_\zeta, b)$, say. By Lemma 3,

$$\begin{aligned} \sum\nolimits_{j=1}^{2} C_j(r, G_\zeta, 0) &\gtreqqless 1 \text{ if } r \gtreqqless -\frac{1}{2}, \\ \sum\nolimits_{j=1}^{2} C_j\left(r, G_\zeta, -\tfrac{1}{2} - r\right) &\lesseqqgtr 1 \text{ if } r \gtreqqless -\frac{1}{2}, \end{aligned}$$

and

$$C_3\left(r, G_\zeta, 0\right) \gtreqless \frac{1}{2} \text{ if } r \gtreqless -\frac{1}{2},$$
$$C_3\left(r, G_\zeta, -\tfrac{1}{2} - r\right) \lesseqgtr \frac{1}{2} \text{ if } r \gtreqless -\frac{1}{2}.$$

The latter follows on writing

$$\begin{aligned} C_3\left(r, G_\zeta, b\right) &= \Pr\left[\frac{u_{it} - (1+r)\zeta_1 + r\zeta_2}{\Delta y_{it-1} + \zeta_1 - \zeta_2} \leq b | \zeta_1 < \zeta_2\right] \Pr\left[\zeta_1 < \zeta_2\right] \\ &+ \Pr\left[\frac{u_{it} - (1+r)\zeta_1 + r\zeta_2}{\Delta y_{it-1} + \zeta_1 - \zeta_2} \leq b | \zeta_1 > \zeta_2\right] \Pr\left[\zeta_1 > \zeta_2\right] \\ &+ \Pr\left[\frac{u_{it} - (1+r)\zeta_1 + r\zeta_2}{\Delta y_{it-1}} \leq b | \zeta_1 = \zeta_2\right] \Pr\left[\zeta_1 = \zeta_2\right]. \end{aligned}$$

Along the same lines,

$$D\left(r, G_\zeta, 0\right) \gtreqless \frac{1}{2} \text{ if } r \gtreqless -\frac{1}{2},$$
$$D\left(r, G_\zeta, -\tfrac{1}{2} - r\right) \lesseqgtr \frac{1}{2} \text{ if } r \gtreqless -\frac{1}{2}.$$

Collecting results, we conclude that for any G_ζ,

$$\Pr\left[\frac{\Delta y_{it}^{\zeta,\varepsilon} - r\Delta y_{it-1}^{\zeta,\varepsilon}}{\Delta y_{it-1}^{\zeta,\varepsilon}} \leq 0\right] \gtreqless \frac{1}{2} \text{ if } r \gtreqless -\frac{1}{2},$$
$$\Pr\left[\frac{\Delta y_{it}^{\zeta,\varepsilon} - r\Delta y_{it-1}^{\zeta,\varepsilon}}{\Delta y_{it-1}^{\zeta,\varepsilon}} \leq -\frac{1}{2} - r\right] \lesseqgtr \frac{1}{2} \text{ if } r \gtreqless -\frac{1}{2},$$

and the desired result follows. ■

Proof of Theorem 4. Let $\mathcal{G}$ be the set of all non-defective univariate cdfs, and let $G_\zeta \in \mathcal{G}$. With r fixed, write $A(r, b)$ as $A(b)$, $B(r, G_\zeta, b)$ as $B(G_\zeta, b)$, and so on. Then

$$
\begin{aligned}
A(b) &= \frac{1}{2}+\frac{1}{\pi}\arctan\frac{b}{\sqrt{1-r^2}},\\
B(G_\zeta,b) &= \int B(z_1;b)dG_\zeta(z_1),\\
C\left(G_\zeta,b\right) &= \int\int C(z_1,z_2;b)dG_\zeta(z_1)dG_\zeta(z_2),\\
D\left(G_\zeta,b\right) &= \int\int\int D(z_1,z_2,z_3;b)dG_\zeta(z_1)dG_\zeta(z_2)dG_\zeta(z_3),
\end{aligned}
$$

where, denoting $b'=r+b$,

$$
\begin{aligned}
B(z_1;b) &= \Pr\left[\frac{\Delta y_{it}+z_1}{\Delta y_{it-1}}\le b'\right]+\Pr\left[\frac{\Delta y_{it}-z_1}{\Delta y_{it-1}+z_1}\le b'\right]\\
&\quad+\Pr\left[\frac{\Delta y_{it}}{\Delta y_{it-1}-z_1}\le b'\right],\\
2C\left(z_1,z_2;b\right) &= \Pr\left[\frac{\Delta y_{it}+z_1-z_2}{\Delta y_{it-1}+z_2}\le b'\right]+\Pr\left[\frac{\Delta y_{it}+z_2-z_1}{\Delta y_{it-1}+z_1}\le b'\right]\\
&\quad+\Pr\left[\frac{\Delta y_{it}+z_1}{\Delta y_{it-1}-z_2}\le b'\right]+\Pr\left[\frac{\Delta y_{it}+z_2}{\Delta y_{it-1}-z_1}\le b'\right]\\
&\quad+\Pr\left[\frac{\Delta y_{it}-z_1}{\Delta y_{it-1}+z_1-z_2}\le b'\right]+\Pr\left[\frac{\Delta y_{it}-z_2}{\Delta y_{it-1}+z_2-z_1}\le b'\right],\\
6D(z_1,z_2,z_3;b) &= \Pr\left[\frac{\Delta y_{it}+z_1-z_2}{\Delta y_{it-1}+z_2-z_3}\le b'\right]+\Pr\left[\frac{\Delta y_{it}+z_1-z_3}{\Delta y_{it-1}+z_3-z_2}\le b'\right]\\
&\quad+\Pr\left[\frac{\Delta y_{it}+z_2-z_1}{\Delta y_{it-1}+z_1-z_3}\le b'\right]+\Pr\left[\frac{\Delta y_{it}+z_2-z_3}{\Delta y_{it-1}+z_3-z_1}\le b'\right]\\
&\quad+\Pr\left[\frac{\Delta y_{it}+z_3-z_1}{\Delta y_{it-1}+z_1-z_2}\le b'\right]+\Pr\left[\frac{\Delta y_{it}+z_3-z_2}{\Delta y_{it-1}+z_2-z_1}\le b'\right].
\end{aligned}
$$

Note that $B(z_1;b)$, $C\left(z_1,z_2;b\right)$, and $D(z_1,z_2,z_3;b)$ are even functions of (z_1,z_2,z_3) and are invariant under permutations of their z-arguments and that $D\left(z_1,z_2,z_3;b\right)$ is invariant under translations of (z_1,z_2,z_3). Let $b(G_\zeta)$ solve

$$
(1-\varepsilon)^3A(b)+(1-\varepsilon)^2\varepsilon B(G_\zeta,b)+(1-\varepsilon)\varepsilon^2C(G_\zeta,b)+\varepsilon^3D(G_\zeta,b)=\frac{1}{2}, \quad (31)
$$

and let $b^* = -\frac{1}{2} - r$. Suppose $r > -\frac{1}{2}$. For all z_1, z_2, z_3,

$$A(b^*) < \frac{1}{2}, \qquad B(z_1; b^*) < \frac{3}{2}, \qquad C(z_1, z_2; b^*) < \frac{3}{2}, \qquad D(z_1, z_2, z_3; b^*) < \frac{1}{2}.$$

So, for all $G_\zeta \in \mathcal{G}$,

$$A(b^*) < \frac{1}{2}, \qquad B(G_\zeta, b^*) < \frac{3}{2}, \qquad C(G_\zeta, b^*) < \frac{3}{2}, \qquad D(G_\zeta, b^*) < \frac{1}{2},$$

which implies

$$\sup_{G_\zeta \in \mathcal{G}} B(G_\zeta, b^*) \leq \frac{3}{2}, \qquad \sup_{G_\zeta \in \mathcal{G}} C(G_\zeta, b^*) \leq \frac{3}{2}, \qquad \sup_{G_\zeta \in \mathcal{G}} D(G_\zeta, b^*) \leq \frac{1}{2}.$$

For all $G_\zeta \in \mathcal{G}$, $b^* < b(G_\zeta) < 0$. Breakdown to 0 occurs if and only if $b(G_\zeta) \downarrow b^*$ as $G_\zeta \to \partial\mathcal{G}$, where $\partial\mathcal{G}$ is the boundary of $\mathcal{G}$ (i.e., $\partial\mathcal{G}$ is the set of defective cdfs). Let $a = \frac{1-\varepsilon}{\varepsilon}$ and

$$S(G_\zeta, b) = a^2(B(G_\zeta, b) - \tfrac{3}{2}) + a(C(G_\zeta, b) - \tfrac{3}{2}) + D(G_\zeta, b) - \tfrac{1}{2}.$$

Then, by (31), breakdown to 0 is possible if and only if

$$\lim_{b \downarrow b^*} \sup_{G_\zeta \in \mathcal{G}} S(G_\zeta, b) \geq a^3 c,$$

where

$$c = \frac{1}{\pi} \arctan \frac{\left|r + \frac{1}{2}\right|}{\sqrt{1 - r^2}} = \left|A(b^*) - \tfrac{1}{2}\right|. \tag{32}$$

Note that $0 \leq c < \frac{1}{2}$ for all $r \in (-1, 0]$. Now consider the sequential limit $G_\zeta \to \partial\mathcal{G}$ followed by $b \downarrow b^*$. For any b satisfying $0 > b > b^*$, as $z \to \infty$ we obtain, recallling $b' = r + b$,

$$\begin{aligned}
B(z;b) &= \Pr\left[\frac{\Delta y_{it}+z}{\Delta y_{it-1}} \le b'\right] + \Pr\left[\frac{\Delta y_{it}-z}{\Delta y_{it-1}+z} \le b'\right] \\
&\quad + \Pr\left[\frac{\Delta y_{it}}{\Delta y_{it-1}-z} \le b'\right] \\
&\to \frac{1}{2}+1+0=\frac{3}{2}, \\
2C\left(z,kz;b\right) &= \Pr\left[\frac{\Delta y_{it}+\left(1-k\right)z}{\Delta y_{it-1}+kz} \le b'\right] + \Pr\left[\frac{\Delta y_{it}+\left(k-1\right)z}{\Delta y_{it-1}+z} \le b'\right] \\
&\quad + \Pr\left[\frac{\Delta y_{it}+z}{\Delta y_{it-1}-kz} \le b'\right] + \Pr\left[\frac{\Delta y_{it}+kz}{\Delta y_{it-1}-z} \le b'\right] \\
&\quad + \Pr\left[\frac{\Delta y_{it}-z}{\Delta y_{it-1}+\left(1-k\right)z} \le b'\right] + \Pr\left[\frac{\Delta y_{it}-kz}{\Delta y_{it-1}+\left(k-1\right)z} \le b'\right] \\
&\to \begin{cases} 4 & \text{if } k \in \{-1,\frac{1}{2},2\}, \\ 3 & \text{otherwise,} \end{cases} \\
6D(0,z,kz;b) &= \Pr\left[\frac{\Delta y_{it}-z}{\Delta y_{it-1}+\left(1-k\right)z} \le b'\right] + \Pr\left[\frac{\Delta y_{it}-kz}{\Delta y_{it-1}+\left(k-1\right)z} \le b'\right] \\
&\quad + \Pr\left[\frac{\Delta y_{it}+z}{\Delta y_{it-1}-kz} \le b'\right] + \Pr\left[\frac{\Delta y_{it}+\left(1-k\right)z}{\Delta y_{it-1}+kz} \le b'\right] \\
&\quad + \Pr\left[\frac{\Delta y_{it}+kz}{\Delta y_{it-1}-z} \le b'\right] + \Pr\left[\frac{\Delta y_{it}+\left(k-1\right)z}{\Delta y_{it-1}+z} \le b'\right] \\
&\to \begin{cases} 4 & \text{if } k = \{-1,\frac{1}{2},2\}, \\ 3 & \text{otherwise.} \end{cases}
\end{aligned}$$

Also note that $D(z,z,z;b) = A(b)$. Letting $b \downarrow b^*$ gives

$$\begin{aligned}
\lim_{b\downarrow b^*}\lim_{z\to\infty} B\left(z;b\right) &= \frac{3}{2}, \\
\lim_{b\downarrow b^*}\lim_{z\to\infty} C\left(z,kz;b\right) &= \begin{cases} 2 & \text{if } k \in \{-1,\frac{1}{2},2\}, \\ \frac{3}{2} & \text{otherwise,} \end{cases} && (33) \\
\lim_{b\downarrow b^*}\lim_{z\to\infty} D\left(0,z,kz;b\right) &= \begin{cases} \frac{2}{3} & \text{if } k = \{-1,\frac{1}{2},2\}, \\ \frac{1}{2} & \text{otherwise.} \end{cases} && (34)
\end{aligned}$$

Thus, in order to maximize $B(G_\zeta, b)$ and $C\left(G_\zeta, b\right)$ as $b \downarrow b^*$, all mass of G_ζ must be taken to $\pm\infty$ as $G_\zeta \to \partial\mathcal{G}$. This does not restrict any possibilities of maximizing $D\left(G_\zeta, b\right)$ (hence, of maximizing $S(G_\zeta, b)$) as $b \downarrow b^*$

because $D(G_\zeta, b)$ is invariant under translations of G_ζ. Further, G_ζ must fall into atoms as $G_\zeta \to \partial\mathcal{G}$. To see this, suppose G_ζ is continuous and does not fall into atoms as $G_\zeta \to \partial\mathcal{G}$; then $\lim_{b\downarrow b^*} \lim_{G_\zeta\to\partial\mathcal{G}} C(G_\zeta, b) = \frac{3}{2}$ and $\lim_{b\downarrow b^*} \lim_{G_\zeta\to\partial\mathcal{G}} D(G_\zeta, b) = \frac{1}{2}$; more generally, if G_ζ has a continuous portion that does not fall into atoms, this can only reduce the maximum values that $\lim_{b\downarrow b^*} \lim_{G_\zeta\to\partial\mathcal{G}} C(G_\zeta, b)$ and $\lim_{b\downarrow b^*} \lim_{G_\zeta\to\partial\mathcal{G}} D(G_\zeta, b)$ can attain. Hence, in view of (33)–(34), whenever breakdown to zero is possible, it can be achieved by some G_ζ defined by $\Pr[\zeta = iz] = p_i$, $i \in \mathbb{Z}_0$, and the path $G_\zeta \to \partial\mathcal{G}$ defined by $z \to \infty$. For such G_ζ,

$$\begin{aligned}
\lim_{b\downarrow b^*} \lim_{z\to\infty} B(G_\zeta, b) &= \frac{3}{2}, \\
\lim_{b\downarrow b^*} \lim_{z\to\infty} C(G_\zeta, b) &= \frac{3}{2} + \sum_{i>0} p_{-i}p_i + \sum_i p_i p_{2i}, \\
\lim_{b\downarrow b^*} \lim_{z\to\infty} D(G_\zeta, b) &= \frac{1}{2} + \sum_{\substack{i<j<k \\ 2j=i+k}} p_i p_j p_k - c\sum_i p_i^3.
\end{aligned}$$

We conclude that, when G_ζ is allowed to vary over the whole set $\mathcal{G}$, breakdown to 0 is possible if and only if

$$\max_P J(P, a, c) \geq a^3 c, \tag{35}$$

where $P = (p_i)_{i\in\mathbb{Z}_0}$, $\sum_i p_i = 1$, $p_i \geq 0$ for all $i \in \mathbb{Z}_0$, and

$$J(P, a, c) = a\sum_{i>0} p_{-i}p_i + a\sum_i p_i p_{2i} + \sum_{\substack{i<j<k \\ 2j=i+k}} p_i p_j p_k - c\sum_i p_i^3. \tag{36}$$

When $-1 < r < -\frac{1}{2}$, we arrive at the same conclusion because now breakdown to 0 is possible if and only if

$$\lim_{b\uparrow b^*} \inf_{G_\zeta\in\mathcal{G}} S(G_\zeta, b) \leq -a^3 c,$$

and we have

$$\begin{aligned}
\lim_{b\uparrow b^*}\lim_{z\to\infty} B\left(z;b\right) &= \frac{3}{2}, \\
\lim_{b\uparrow b^*}\lim_{z\to\infty} C\left(z,kz;b\right) &= \begin{cases} 1 & \text{if } k\in\{-1,\frac{1}{2},2\}, \\ \frac{3}{2} & \text{otherwise,}\end{cases} \\
\lim_{b\uparrow b^*}\lim_{z\to\infty} D\left(0,z,kz;b\right) &= \begin{cases} \frac{1}{3} & \text{if } k\in\{-1,\frac{1}{2},2\}, \\ \frac{1}{2} & \text{otherwise.}\end{cases}
\end{aligned}$$

When G_ζ is defined by $\Pr[\zeta = z] = \Pr[\zeta = -z] = \Pr[\zeta = 2z] = \Pr[\zeta = -2z] = \frac{1}{4}$, we consider P with $p_1 = p_{-1} = p_2 = p_{-2} = \frac{1}{4}$, giving $J(P,a,c) = \frac{a}{4} - \frac{c}{16}$ and the breakdown condition $\frac{a}{4} - \frac{c}{16} \geq a^3c$, which is (7). Because $c < \frac{1}{2}$, this condition is always satisfied when $a = \frac{1}{2}$. The breakdown point under this type of contamination is the smallest value of ε and thus the largest value of a for which $\frac{a}{4} - \frac{c}{16} \geq a^3c$. Now let G_ζ be any symmetric four-point distribution. Then the maximum of $J(P,a,c)$ is the largest of

$$\max_{p_1+p_2=\frac{1}{2}} \left\{a\left(p_1^2+p_2^2\right) + 2ap_1p_2 - 2c\left(p_1^3+p_2^3\right)\right\} = \frac{a}{4} - \frac{c}{16}$$

and

$$\max_{p_1+p_3=\frac{1}{2}} \left\{a\left(p_1^2+p_3^2\right) + 2p_1^2p_3 - 2c\left(p_1^3+p_3^3\right)\right\}.$$

The former maximum dominates the latter whenever $a \geq \frac{1}{2}$. Hence, the breakdown point under symmetric four-point AO is the smallest ε solving (7). ■

Proof of Theorem 5. Suppose $r > -\frac{1}{2}$. From the proof of Theorem 4, breakdown to zero is possible if and only if $\max_P J(P,a,c) \geq a^3c$. Clearly,

$$\max_P J(P,a,c) \leq a\max_P \alpha + \max_P \beta - c\min_P \gamma,$$

where

$$\alpha = \sum_{i>0} p_{-i}p_i + \sum_i p_ip_{2i}, \qquad \beta = \sum_{\substack{i<j<k \\ 2j=i+k}} p_ip_jp_k, \qquad \gamma = \sum_i p_i^3.$$

We have $\max_P \alpha = \frac{1}{4}$ and $\min_P \gamma = 0$. Regarding β, each set of three points $\{i, j, k\}$ where one point is equidistant from the other two contributes a term $p_i p_j p_k$. Hence, $\beta \leq \beta'$ where β' is the same as β except that all n points are placed at equal intervals on a circle instead of a line. Maximizing β' for given n is a symmetric problem, so the solution will involve equal probabilities for all points or equal probabilities for some points and zero probabilities for the other points. In the latter case, the problem is reduced to a smaller n. For $n = 2k + 1$ with $k \geq 1$, each point has k pairs of points from which it is equidistant, so, when the probabilities are equal, $\beta' = \frac{k}{(2k+1)^2} - \frac{2l}{(2k+1)^3}$, where $l = 1$ if n is divisible by 3 and $l = 0$ otherwise. Similarly, for $n = 2k$ with $k \geq 2$, when the probabilities are equal, $\beta' = \frac{k-1}{4k^2} - \frac{2l}{8k^3}$. The largest value of β' is attained when $n = 5$, giving $\beta' = \frac{2}{25} \geq \max_P \beta$. Hence, breakdown requires $\frac{a}{4} + \frac{2}{25} \geq a^3 c$. When $r < -\frac{1}{2}$, breakdown requires $\frac{a}{4} + \frac{2}{25} \leq a^3 c$. In any case, the breakdown point must be at least as large as the smallest $\varepsilon > 0$ solving $\frac{a}{4} + \frac{2}{25} = a^3 c$ or, equivalently, (8). ■

Bias of $\hat{\rho}$ under patched AO with point-mass distribution. As before, $\text{Bias}(\widehat{\rho}; \rho, \zeta, \varepsilon, k) = 2b$, where now b solves

$$\begin{aligned}&\left(p^2 (1-p)^k - 4p (1-p)^k + 1\right) A(r,b) + p(1-p)^k (2-p) B_{1,3}(r,\zeta,b) \\ &+ p^2 (1-p)^k B_2(r,\zeta,b) = \frac{1}{2}, \qquad (37)\end{aligned}$$

with

$$\begin{aligned} A(r,b) &= \Pr\left[\frac{\Delta y_{it}}{\Delta y_{it-1}} \leq r + b\right], \\ B_{1,3}(r,\zeta,b) &= \Pr\left[\frac{\Delta y_{it} + \zeta}{\Delta y_{it-1}} \leq r + b\right] + \Pr\left[\frac{\Delta y_{it}}{\Delta y_{it-1} - \zeta} \leq r + b\right], \\ B_2(r,\zeta,b) &= \Pr\left[\frac{\Delta y_{it} + \zeta}{\Delta y_{it-1} - \zeta} \leq r + b\right], \end{aligned}$$

and $(1-p)^k = 1 - \varepsilon$. Using the same method as we used for independent AO, we can compute b. ■

Proof of Theorem 6. Rewrite the left-hand side of (37) as

$$\begin{aligned} V(r,\zeta,b,k,p) &= \left(p^2(1-p)^k - 4p(1-p)^k + 1\right) A(r,b) \\ &\quad +2p(1-p)^{k+1} B_{1,3}(r,\zeta,b) + p^2(1-p)^k B(b,\zeta,r), \end{aligned}$$

where

$$\begin{aligned} B(r,\zeta,b) &= \Pr\left[\frac{\Delta y_{it}+\zeta}{\Delta y_{it-1}} \le r+b\right] + \Pr\left[\frac{\Delta y_{it}}{\Delta y_{it-1}-\zeta} \le r+b\right] \\ &\quad + \Pr\left[\frac{\Delta y_{it}+\zeta}{\Delta y_{it-1}-\zeta} \le r+b\right]. \end{aligned}$$

$B(r,\zeta,b)$ is the same as in the proof of Theorem 1, so

$$B(r,\zeta,0) \gtreqless \frac{3}{2} \text{ if } r \gtreqless -\frac{1}{2}.$$

In addition, $B_{1,3}(r,\zeta,0) \le 1$, with equality if and only if $r = 0$, since $\Pr\left[\frac{\Delta y_{it}+\zeta}{\Delta y_{it-1}} \le r\right] = \frac{1}{2}$ and $\Pr\left[\frac{\Delta y_{it}}{\Delta y_{it-1}-\zeta} \le r\right] \le \frac{1}{2}$, with equality if and only if $r = 0$. Consequently, if $\rho \le 0$,

$$V(r,\zeta,0,k,p) < \frac{1}{2},$$

and b, solving $V(r,\zeta,b,k,p) = \frac{1}{2}$, is positive. If $0 < \rho < 1$, then $B(r,\zeta,0) > \frac{3}{2}$ and $B_{1,3}(r,\zeta,0) < 1$, making the sign of the bias dependent on the values of ρ, ζ, k, and ε. ■

Proof of Theorem 7. The influence function is found along the lines of the proof of Theorem 2. Now,

$$\mathrm{IF}(\widehat{\rho};\rho,\zeta,k) = 2\left.\frac{\partial b(r,\zeta,\varepsilon,k)}{\partial\varepsilon}\right|_{\varepsilon=0},$$

where b solves (37). Differentiating (37) and evaluating at $p = 0$ gives

$$\begin{aligned} \left.\frac{\partial b(r,\zeta,\varepsilon,k)}{\partial p}\right|_{p=0} &= \frac{4A(r,0) - 2B_{1,3}(r,\zeta,0)}{A_b(r,0)} \\ &= (2 - 2B_{1,3}(r,\zeta,0))\,\pi\sqrt{1-r^2}. \end{aligned}$$

Since

$$\frac{\partial b(r,\zeta,\varepsilon,k)}{\partial \varepsilon} = \frac{\partial b(r,\zeta,\varepsilon,k)}{\partial p}\frac{\partial p}{\partial \varepsilon} = \frac{\partial b(r,\zeta,\varepsilon,k)}{\partial p}\frac{1}{k\left(1-p\right)^{k-1}},$$

we obtain

$$\mathrm{IF}(\widehat{\rho};\rho,\zeta,k) = \frac{4-4B_{1,3}\left(r,\zeta,0\right)}{k}\pi\sqrt{1-r^2},$$

where $B_{1,3}\left(r,\zeta,0\right)$ can be written as

$$\begin{aligned} B_{1,3}\left(r,\zeta,0\right) &= 1+\frac{1}{2}\left[\Phi\left(\frac{\sqrt{1+r}\zeta}{\sigma}\right)-\Phi\left(\frac{-\sqrt{1+r}\zeta}{\sigma}\right)\right] \\ &\quad \times\left[\Phi\left(\frac{r\zeta}{\sigma\sqrt{1-r}}\right)-\Phi\left(\frac{-r\zeta}{\sigma\sqrt{1-r}}\right)\right]. \end{aligned}$$

The expression for $\mathrm{IF}(\widehat{\rho};\rho,\zeta,k)$ follows. Clearly, $\mathrm{IF}(\widehat{\rho};\rho,\zeta,k) \geq 0$, with equality if and only if $\zeta = 0$ or $r = 0$. So, when $\rho = 1$, $\mathrm{GES}(\widehat{\rho};\rho,k) = 0$. When $\rho < 1$, $\mathrm{IF}(\widehat{\rho};\rho,\zeta,k)$ increases in $|\zeta|$, and, therefore, $\mathrm{GES}(\widehat{\rho};\rho,k) = \lim_{\zeta\to\infty}\mathrm{IF}(\widehat{\rho};\rho,\zeta,k) = \frac{2}{k}\pi\sqrt{1-r^2}$. ■

Proof of Theorem 8. Denote the left-hand side of (37) as $V\left(r,\zeta,b,k,p\right)$, and let $\zeta\to\infty$ without loss of generality since $\mathrm{Bias}(\widehat{\rho};\rho,\zeta,\varepsilon,k) = \mathrm{Bias}(\widehat{\rho};\rho,-\zeta,\varepsilon,k)$. As $\zeta\to\infty$, $\widehat{\rho}$ breaks down to 1 if and only if

$$\lim_{b\uparrow -r}\lim_{\zeta\to\infty} V\left(r,\zeta,b,k,p\right) \leq \frac{1}{2}.$$

Now

$$\begin{aligned} \lim_{b\uparrow -r}\lim_{\zeta\to\infty} B_{1,3}\left(r,\zeta,b\right) &= \frac{1}{2}, \\ \lim_{b\uparrow -r}\lim_{\zeta\to\infty} B_{2}\left(r,\zeta,b\right) &= 1, \\ \lim_{b\uparrow -r} A\left(r,b\right) &= \frac{1}{2}+\frac{1}{\pi}\arctan\frac{-r}{\sqrt{1-r^2}}. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{b\uparrow -r}\lim_{\zeta\to\infty} V\left(r,\zeta,b,k,p\right) &= \left(p^2\left(1-p\right)^k-4p\left(1-p\right)^k+1\right)\frac{1}{\pi}\arctan\frac{-r}{\sqrt{1-r^2}} \\ &\quad -p\left(1-p\right)^{k+1}+\frac{1}{2}, \end{aligned}$$

and the result follows. ■

Bias of $\hat{\rho}$ under patched AO with arbitrary distribution. We have $\text{Bias}(\widehat{\rho};\rho,G_\zeta,\varepsilon,k)=2b$, with b solving

$$\begin{aligned}\frac{1}{2} &= q^2\left(1-q^{k-2}+q^k\right)A\left(r,b\right)+pq^{k+1}B_{1,3}\left(r,G_\zeta,b\right)+pq^kB_1\left(r,G_\zeta,b\right)\\ &\quad +pq^{k+1}B_3\left(r,G_\zeta,b\right)+p^2q^kC_{1,2}\left(r,G_\zeta,b\right)+p^2\left(1-q^k\right)D\left(r,G_\zeta,b\right)\\ &\quad +\left(1-q^{k-1}\right)pqE_1\left(r,G_\zeta,b\right)+\left(1-q^k\right)pqE_2\left(r,G_\zeta,b\right), \qquad (38)\end{aligned}$$

where $q=1-p$, $A\left(r,b\right)$ is as before, and

$$\begin{aligned}B_1\left(r,G_\zeta,b\right) &= \int \Pr\left[\frac{\Delta y_{it}+z}{\Delta y_{it-1}}\le r+b\right]dG_\zeta\left(z\right),\\ B_3\left(r,G_\zeta,b\right) &= \int \Pr\left[\frac{\Delta y_{it}}{\Delta y_{it-1}-z}\le r+b\right]dG_\zeta\left(z\right),\\ B_{1,3}\left(r,G_\zeta,b\right) &= B_1\left(r,G_\zeta,b\right)+B_3\left(r,G_\zeta,b\right),\\ C_{1,2}\left(r,G_\zeta,b\right) &= \int\int\left(\Pr\left[\frac{\Delta y_{it}+z_1-z_2}{\Delta y_{it-1}+z_2}\le r+b\right]+\right.\\ &\quad \left.\Pr\left[\frac{\Delta y_{it}+z_1}{\Delta y_{it-1}-z_2}\le r+b\right]\right)dG_\zeta\left(z_1\right)dG_\zeta\left(z_2\right),\\ D\left(r,G_\zeta,b\right) &= \int\int\int \Pr\left[\frac{\Delta y_{it}+z_1-z_2}{\Delta y_{it-1}+z_2-z_3}\le r+b\right]dG_\zeta\left(z_1\right)dG_\zeta\left(z_2\right)dG_\zeta\left(z_3\right),\\ E_1\left(r,G_\zeta,b\right) &= \int\int \Pr\left[\frac{\Delta y_{it}+z_2-z_1}{\Delta y_{it-1}}\le r+b\right]dG_\zeta\left(z_1\right)dG_\zeta\left(z_2\right),\\ E_2\left(r,G_\zeta,b\right) &= \int\int \Pr\left[\frac{\Delta y_{it}}{\Delta y_{it-1}+z_2-z_1}\le r+b\right]dG_\zeta\left(z_1\right)dG_\zeta\left(z_2\right).\end{aligned}$$

■

Proof of Theorem 9. In view of Theorem 6, we only need to prove that $\text{Bias}(\widehat{\rho};\rho,G_\zeta,k,\varepsilon)>0$ if $\rho\le 0$. Let $V\left(r,G_\zeta,b,k,p\right)$ denote the right-hand side of (38). The proof is complete if we show that $V\left(r,\zeta,0,k,p\right)<\frac{1}{2}$ when $r\le -\frac{1}{2}$. With $r\le -\frac{1}{2}$, we have $A\left(r,0\right)=\frac{1}{2}$, $B_1\left(r,G_\zeta,0\right)=\frac{1}{2}$, $B_3\left(r,G_\zeta,0\right)<\frac{1}{2}$, $C_{1,2}\left(r,G_\zeta,0\right)<1$, $D\left(r,G_\zeta,b\right)<\frac{1}{2}$, $E_1\left(r,G_\zeta,0\right)=\frac{1}{2}$, and $E_2\left(r,G_\zeta,0\right)<\frac{1}{2}$. It follows that $V\left(r,G_\zeta,0,k,p\right)<\frac{1}{2}$. ■

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