Consistent Estimation with Weak Instruments in Panel Data*

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Abstract

This note analyzes the asymptotic distribution for instrumental variables regression for panel data when the available instruments are weak. We show that consistency can be established in panel data.

Key Words: *Weak Instrument; Two Stage Least Squares; Panel Data; Concentration Parameter.*

1 Motivation and Results

In recent years, economists have been concerned with the problem of weak instruments or partial identification, see Stock, Wright and Yogo (2002) for an excellent survey. Economists found that the first stage F statistic in the two stage least squares (2SLS) regression is often low, say, less than 10. In this case, the usual asymptotic normal approximations can be quite poor, even if the number of observations is large. To provide better asymptotic approximations, Staiger and Stock (1997) derive the weak-instrument asymptotics for instrumental variables estimators. Staiger and Stock show that the 2SLS is inconsistent (i.e., converges to a random variable) and has a nonstandard limiting distribution. In this note we study the asymptotics of 2SLS and k-class estimator with weak instruments in the panel set-up. We show that the consistency of 2SLS and k-class estimator can be established in panel data. It is known that in the cross-sectional data, when the concentration parameter stays constant as the sample size grows, the signal of the model is too weak comparing to the noise. Hence the model is weakly identified, e.g., the 2SLS converges to a random variable hence inconsistent However, in the panel set-up, if time series dimension is large, the weak signal can be strengthened by repeating regression across the time series dimension. It is, in spirit, similar to the argument of establishing the consistency in the panel spurious regression, e.g., Phillips and Moon (1999) and Kao (1999). In this note we use $(n,T) \overset{seq}{\to} \infty$ to denote the sequential limit, i.e., $n \to \infty$ followed by $T \to \infty$.

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Consider the following panel IV regression model with endogenous regressors

$$y_t = Y_t\beta + X_t\gamma + u_t \tag{1}$$

and

$$Y_t = Z_t\Pi + X_t\Phi + V_t \tag{2}$$

$t = 1, 2, \cdots, T$, where y_t and Y_t are $n \times 1$ vector and $n \times L$ vector of endogenous variables, X_t is a $n \times K_1$ matrix of K_1 exogenous regressors, Z_t is a $n \times K_2$ matrix of K_2 instruments, and β, γ, Π and Φ are unknown parameters. The errors $(u_t, V_t')'$ are $n \times 1$ vector and $n \times L$ matrix of error terms respectively. Let u_{it} and v_{it} be ith element of u_t and v_t respectively. The errors $(u_t, V_t)'$ are assumed to be i.i.d. $N(0, \Sigma)$, where the elements of Σ are σ_{uu}, Σ_{Vu} and Σ_{VV}. Let $\bar{Z} = [X, Z]$ and let $Q = E\bar{Z}_{it}'\bar{Z}_{it}$, partitioned so that $EX_{it}X_{it}' = Q_{XX}$, $EX_{it}Z_{it}' = Q_{XZ}$ and $EZ_{it}Z_{it}' = Q_{ZZ}$. Also let $\rho = \Sigma_{VV}^{-1/2\prime}\Sigma_{Vu}\sigma_{uu}^{-1/2}$. It is assumed throughout that $E\bar{Z}_{it}(u_{it}, V_{it}') = 0$ for all i and t. This i.i.d. assumption for the errors can be relaxed to allow weak dependence across time series and cross-section at the expense of complicated notations and will be studied in a different paper. Equation (1) is the structural equation and β is the parameter of interest. The reduced-form equation (2) relates the endogenous regressors to the instruments.

The following assumptions are borrowed from Staiger and Stock (1997).

Assumption 1 *For a given t, as $n \to \infty$, we have*

1. $\left(\frac{1}{n}u_t'u_t, \frac{1}{n}V_t'u_t, \frac{1}{n}V_t'V_t\right) \xrightarrow{p} (\sigma_{uu}, \Sigma_{Vu}, \Sigma_{VV})$;
2. $\frac{1}{n}\bar{Z}_t'\bar{Z}_t \xrightarrow{p} Q$;
3. $\left(\frac{1}{\sqrt{n}}X_t'u_t, \frac{1}{\sqrt{n}}Z_t'u_t, \frac{1}{\sqrt{n}}X_t'V_t, \frac{1}{\sqrt{n}}Z_t'V_t\right) \xrightarrow{d} (\Psi_{Xu}, \Psi_{Zu}, \Psi_{XV}, \Psi_{ZV})$, *where* $\Psi = \left(\Psi_{Xu}', \Psi_{Zu}', vec(\Psi_{XV})', vec(\Psi_{ZV})'\right)'$ *is distributed as* $N(0, \Sigma \otimes Q)$.

Define $z_u = \Omega^{-1/2\prime}\left(\Psi_{Zu} - Q_{ZX}Q_{XX}^{-1}\Psi_{Xu}\right)\sigma_{uu}^{-1/2}$ and $z_V = \Omega^{-1/2\prime}\left(\Psi_{ZV} - Q_{ZX}Q_{XX}^{-1}\Psi_{XV}\right)\Sigma_{VV}^{-1/2}$, where $\Omega = Q_{ZZ} - Q_{ZX}Q_{XX}^{-1}Q_{XZ}$. The random variable $\left[z_u', vec(z_V)'\right]'$ is distributed as $N\left(0, \bar{\Sigma} \otimes I_{K_2}\right)$, where $\bar{\Sigma}$ is the $(n+1)T \times (n+1)T$ matrix with $\bar{\Sigma}_{11} = I_T$, $\bar{\Sigma}_{22} = I_{nT}$, $\bar{\Sigma}_{12} = \rho'$ and $\bar{\Sigma}_{21} = \rho$, where $\bar{\Sigma}$ is partitioned conformably with Σ and I_{nT} denotes the nT-dimension identity matrix.

We use the following Assumption 2 to model Π as local to zero. Note that it is not the standard "Pitman drift" parameterization in the literature, but a sequence in which the first stage parameter, Π, lies outside $C/\sqrt{nT}$ neighborhood[1]. Our justification for making this assumption is that we treat the time dimension

[1] We thank two referees for pointing this out for us.

as a devise of replicating the experiment (i.e., cross-sectional 2SLS regression) so that the model can be identified, i.e., consistency of 2SLS can be recovered, as the time dimension increases.

Assumption 2 $\Pi = C/\sqrt{n}$, *where* C *is a* $k \times 1$ *constant matrix.*

Let $P_W = W\left(W'W\right)^{-1}W'$ and $M_W = I - P_W$ where W is a general $a \times b$ matrix with $a \geq b$. Let "$\perp$" denote the residuals from the projection on X_t, so $y_t^{\perp} = M_{X_t}y_t$, $Z_t^{\perp} = M_{X_t}Z_t$ and $Y_t^{\perp} = M_{X_t}Y_t$. The strength of instrument can be measured by the concentration parameter $\lambda_t'\lambda_t$, which is defined as

$$\lambda_t'\lambda_t = \Sigma_{VV}^{-1/2\prime}\Pi' Z_t^{\perp\prime} Z_t^{\perp}\Pi\Sigma_{VV}^{-1/2}.$$

Define $\lambda = \Omega^{1/2}C\Sigma_{VV}^{-1/2}$. Under Assumption 2 we obtain

$$\lambda_t'\lambda_t = \Sigma_{VV}^{-1/2\prime}C'\left(\frac{1}{n}Z_t^{\perp\prime}Z_t^{\perp}\right)C\Sigma_{VV}^{-1/2} \xrightarrow{p} \Sigma_{VV}^{-1/2\prime}C'\Omega C\Sigma_{VV}^{-1/2} = \lambda'\lambda \tag{3}$$

as $n \to \infty$ where we assume for a give t,

$$\frac{1}{n}Z_t^{\perp\prime}Z_t^{\perp} = \frac{1}{n}Z_t'Z_t - \frac{1}{n}Z_t'X_t\left(\frac{1}{n}X_t'X_t\right)^{-1}\frac{1}{n}X_t'Z_t \xrightarrow{p} Q_{ZZ} - Q_{ZX}Q_{XX}^{-1}Q_{XZ} = \Omega.$$

In matrix form, equations (1) and (2) can be rewritten as

$$y = Y\beta + X\gamma + u \tag{4}$$

and

$$Y = Z\Pi + X\Phi + V. \tag{5}$$

If Σ_{Vu} is nonzero, Y is endogenous. It is easy to show that the OLS estimator of β is inconsistent, i.e., $\hat{\beta}_{OLS} \xrightarrow{p} \beta + \Sigma_{VV}^{-1}\Sigma_{Vu}$.

2 Asymptotics

We first discuss asymptotics of the 2SLS estimator in a special case when $L = 1$ and $K_1 = 0$, i.e., without regressor X. The 2SLS estimator is defined as

$$\widehat{\beta}_{2SLS} = \frac{Y'P_Z y}{Y'P_Z Y}.$$

Note

$$\widehat{\beta}_{2SLS} - \beta = \frac{Y'P_Z u}{Y'P_Z Y}.$$

By Assumption (1), we can easily obtain as $(n,T) \overset{seq}{\to} \infty$

$$\frac{1}{Tn} Z'Z = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{1}{n} Z_t' Z_t \right) \xrightarrow{p} Q_{ZZ} \tag{6}$$

and

$$\frac{1}{T\sqrt{n}} Z'Y = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{1}{\sqrt{n}} Z_t' Y_t \right) = \frac{1}{T} \sum_{t=1}^{T} \left(Q_{ZZ} C + \Psi_{ZV} \right) \xrightarrow{p} Q_{ZZ} C$$

with $\frac{1}{T} \sum_{t=1}^{T} \Psi_{ZV} \xrightarrow{p} 0$. It follows that

$$\frac{1}{T} Y' P_Z Y = \left(\frac{1}{T\sqrt{n}} Y'Z \right) \left(\frac{1}{Tn} ZZ \right)^{-1} \left(\frac{1}{T\sqrt{n}} Z'Y \right) \xrightarrow{p} \left(Q_{ZZ} C \right)' Q_{ZZ}^{-1} Q_{ZZ} C = C' Q_{ZZ}' Q_{ZZ}^{-1} Q_{ZZ} C$$

as $(n,T) \overset{seq}{\to} \infty$. Similarly, as $(n,T) \overset{seq}{\to} \infty$

$$\frac{1}{T\sqrt{n}} Z'u = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{1}{\sqrt{n}} Z_t' u_t \right) \xrightarrow{p} 0 \tag{7}$$

and

$$\frac{1}{\sqrt{Tn}} Z'u = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(\frac{1}{\sqrt{n}} Z_t' u_t \right) \xrightarrow{d} \Psi_{Zu}$$

with $\frac{1}{\sqrt{n}} Z_t' u_t \xrightarrow{d} \Psi_{Zu}$ for a given t as $n \to \infty$.

Recall that

$$\frac{1}{\sqrt{n}} Z_t' Y_t = \frac{1}{\sqrt{n}} Z_t' Z_t \Pi + \frac{1}{\sqrt{n}} Z_t' V_t = \left(\frac{1}{n} Z_t' Z_t \right) C + \frac{1}{\sqrt{n}} Z_t' V_t \xrightarrow{d} Q_{ZZ} C + \Psi_{ZV}$$

as $n \to \infty$. It is easy to see that

$$\frac{1}{T} \sum_{t=1}^{T} \frac{1}{\sqrt{n}} Z_t' Y_t \xrightarrow{p} Q_{ZZ} C$$

since $\frac{1}{T} \sum_{t=1}^{T} \Psi_{ZV} = o_p(1)$. Hence

$$\frac{1}{T} Y' P_Z u = \left(\frac{1}{T\sqrt{n}} Y'Z \right) \left(\frac{1}{Tn} ZZ \right)^{-1} \left(\frac{1}{T\sqrt{n}} Z'u \right) \xrightarrow{p} C' Q_{ZZ}' Q_{ZZ} \times 0 = o_p(1).$$

It follows that

$$\widehat{\beta}_{2SLS} \xrightarrow{p} \beta$$

as $(n,T) \overset{seq}{\to} \infty$. Hence $\widehat{\beta}_{2SLS}$ is consistent. Next, we show that $\widehat{\beta}_{2SLS}$ is $\sqrt{T}$ consistent and normally distributed.

$$\begin{aligned}
\sqrt{T}\left(\widehat{\beta}_{2SLS}-\beta\right) &= \frac{\left(\frac{1}{T\sqrt{n}}Y'Z\right)\left(\frac{1}{Tn}ZZ\right)^{-1}\left(\frac{1}{\sqrt{Tn}}Z'u\right)}{\left(\frac{1}{T\sqrt{n}}Y'Z\right)\left(\frac{1}{Tn}ZZ\right)^{-1}\left(\frac{1}{T\sqrt{n}}Z'Y\right)} \xrightarrow{d} \frac{C'Q_{ZZ}Q_{ZZ}^{-1}\Psi_{Zu}}{C'Q_{ZZ}Q_{ZZ}^{-1}Q_{ZZ}C} \\
&= \frac{C'\Psi_{Zu}}{C'Q_{ZZ}C} \sim N\left(0, \frac{C'}{C'Q_{ZZ}C}\left(\sigma_{uu}Q_{ZZ}\right)\frac{C}{C'Q_{ZZ}C}\right) \\
&= N\left(0, \frac{\sigma_{uu}}{C'Q_{ZZ}C}\right) \sim N\left(0, \sigma_{uu}\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)^{-1}\right)
\end{aligned}$$

as $(n,T)\overset{seq}{\to}\infty$, because $\Psi_{Zu}\sim N\left(0,\sigma_{uu}Q_{ZZ}\right)$ and $\lambda'\lambda=\Sigma_{VV}^{-1/2\prime}C'Q_{ZZ}C\Sigma_{VV}^{-1/2}$. Then we have the following theorem.[2]

Theorem 1 *Under Assumptions 1 - 2, as* $(n,T)\overset{seq}{\to}\infty$*, we have*

1. $\widehat{\beta}_{2SLS}\xrightarrow{p}\beta$.
2. $\sqrt{T}\left(\widehat{\beta}_{2SLS}-\beta\right)\xrightarrow{d}N\left(0,\sigma_{uu}\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)^{-1}\right)$.

We now generalize the results to the k-class estimator with regressor X. Premultiplying M_X to equations (4) and (5), we have

$$y^{\perp}=Y^{\perp}\beta+u^{\perp}$$

and

$$Y^{\perp}=Z^{\perp}\Pi+V^{\perp}$$

where $y^{\perp}=M_Xy$, $Z^{\perp}=M_XZ, Y^{\perp}=M_XY$. The the k-class estimator of β is

$$\widehat{\beta}(k)=\left[Y^{\perp\prime}\left(I-kM_{Z^{\perp}}\right)Y^{\perp}\right]^{-1}\left[Y^{\perp\prime}\left(I-kM_{Z^{\perp}}\right)y^{\perp}\right].$$

Note that the 2SLS estimator is a special case of k-class estimator when $k=1$. Next we show that the asymptotic property of k-class estimator.

Theorem 2 *Under Assumptions 1 - 2. As* $(n,T)\overset{seq}{\to}\infty$ *joint with* $\kappa_{Tn}=\sqrt{T}n\left(k-1\right)=O_p\left(1\right)$*, we have*

1. $\widehat{\beta}(k)\xrightarrow{p}\beta$.
2. $\sqrt{T}\left[\hat{\beta}(k)-\beta\right]+\theta_{Tn}\xrightarrow{d}N\left(0,\sigma_{uu}\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)^{-1}\right)$,
   where $\theta_{Tn}=\left[\frac{1}{T}Y^{\perp\prime}\left(I-kM_{Z^{\perp}}\right)Y^{\perp}\right]^{-1}\kappa_{Tn}\Sigma_{Vu}$.

See the Appendix for the proof.

In particular, when $k=1$, hence $\kappa_{Tn}=0$, the k-class estimator is reduced to the 2SLS estimator and indeed has the same asymptotic distribution presented in Theorem 1.

[2] We noticed that results in Theorem 1 are also mentioned in Cai and Fang (2008) while we were revising this note.

References

[1] Cai, Z., and Y. Fang (2008) Reducing the Asymptotic Bias of Weak Instruments Estimation Using Independently Repeated Cross-Sectional Information. *Working paper,* Xiamen University.

[2] Kao, C. (1999) Spurious Regression and Residual-Based Tests for Cointegration in Panel Data. *Journal of Econometrics* 90, 1-44.

[3] Phillips, P. C. B., and H. R. Moon (1999) Linear Regression Limit Theory for Nonstationary Panel Data. *Econometrica* 67, 1057-1111.

[4] Staiger, D., and J. H. Stock (1997) Instrumental Variables Regression with Weak Instruments. *Econometrica* 65, 557-587.

[5] Stock, H. J., H. J. Wright, and M. Yogo (2002) A Survey of Weak Instruments and Weak Identification in Generalized Method of Moments. *Journal of Business and Economic Statistics* 20, 518-529.

Appendix

A Lemmas

We first present the following lemma:

Lemma 1 *Under Assumptions 1 - 2, as* $(n,T) \overset{seq}{\to} \infty$*,*

1. $\frac{1}{Tn}u'u \xrightarrow{p} \sigma_{uu}$, $\frac{1}{Tn}X'u \xrightarrow{p} 0$, $\frac{1}{Tn}Y'u \xrightarrow{p} \Sigma_{Vu}$, $\frac{1}{Tn}Y'X \xrightarrow{p} 0$, $\frac{1}{Tn}Y'Y \xrightarrow{p} \Sigma_{VV}$, $\frac{1}{Tn}Z'Z \xrightarrow{p} Q_{ZZ}$, $\frac{1}{Tn}Z'X \xrightarrow{p} Q_{ZX}$ and $\frac{1}{Tn}X'X \xrightarrow{p} Q_{XX}$.

2. $\frac{1}{Tn}u^{\perp\prime}u^{\perp} \xrightarrow{p} \sigma_{uu}$, $\frac{1}{Tn}Y^{\perp\prime}u^{\perp} \xrightarrow{p} \Sigma_{Vu}$, $\frac{1}{Tn}Y^{\perp\prime}Y^{\perp} \xrightarrow{p} \Sigma_{VV}$ and $\frac{1}{Tn}Z^{\perp\prime}Z^{\perp} \xrightarrow{p} Q_{ZZ} - Q_{ZX}Q_{XX}^{-1}Q_{XZ} = \Omega$.

3. $\frac{1}{\sqrt{Tn}}Z^{\perp\prime}u^{\perp} \xrightarrow{d} \Psi_{Zu} - Q_{ZX}Q_{XX}^{-1}\Psi_{Xu}$ and $P_{Z^{\perp}}^{1/2}u^{\perp} \xrightarrow{d} \sigma_{uu}^{1/2}z_u$.

4. $\frac{1}{T\sqrt{n}}Z^{\perp\prime}Y^{\perp} \xrightarrow{p} \Omega C$ and $\frac{1}{\sqrt{T}}P_{Z^{\perp}}^{1/2}Y^{\perp} \xrightarrow{p} \lambda\Sigma_{VV}^{1/2}$.

5. $u^{\perp\prime}P_{Z^{\perp}}u^{\perp} \xrightarrow{d} \sigma_{uu}z_u'z_u$, $\frac{1}{\sqrt{T}}Y^{\perp\prime}P_{Z^{\perp}}u^{\perp} \xrightarrow{d} \sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}\lambda'z_u$, $\frac{1}{T}Y^{\perp\prime}P_{Z^{\perp}}u^{\perp} \xrightarrow{p} 0$ and $\frac{1}{T}Y^{\perp\prime}P_{Z^{\perp}}Y^{\perp} \xrightarrow{p} \Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}$.

Proof. (1) is easy. Consider (2). From Lemma 1, we have

$$\begin{aligned}\frac{1}{Tn}u^{\perp\prime}u^{\perp} &= \frac{1}{Tn}u'M_Xu \\ &= \frac{1}{Tn}u'u-\left(\frac{1}{Tn}u'X\right)\left(\frac{1}{Tn}X'X\right)^{-1}\left(\frac{1}{Tn}X'u\right)\xrightarrow{p}\sigma_{uu},\end{aligned}$$

$$\begin{aligned}\frac{1}{Tn}Y^{\perp\prime}u^{\perp} &= \frac{1}{Tn}Y'M_Xu \\ &= \frac{1}{Tn}Y'u-\left(\frac{1}{Tn}Y'X\right)\left(\frac{1}{Tn}X'X\right)^{-1}\left(\frac{1}{Tn}X'u\right)\xrightarrow{p}\Sigma_{Vu},\end{aligned}$$

$$\begin{aligned}\frac{1}{Tn}Y^{\perp\prime}Y^{\perp} &= \frac{1}{Tn}Y'M_XY \\ &= \frac{1}{Tn}Y'Y-\left(\frac{1}{Tn}Y'X\right)\left(\frac{1}{Tn}X'X\right)^{-1}\left(\frac{1}{Tn}X'Y\right)\xrightarrow{p}\Sigma_{VV}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{Tn}Z^{\perp\prime}Z^{\perp} &= \frac{1}{Tn}Z'M_XZ \\ &= \frac{1}{Tn}Z'Z-\left(\frac{1}{Tn}Z'X\right)\left(\frac{1}{Tn}X'X\right)^{-1}\left(\frac{1}{Tn}X'Z\right)\xrightarrow{p}Q_{ZZ}-Q_{ZX}Q_{XX}^{-1}Q_{XZ}=\Omega\end{aligned}$$

as $(n,T)\overset{seq}{\to}\infty$.

Consider (3). Note that

$$\begin{aligned}\frac{1}{\sqrt{Tn}}Z^{\perp\prime}u^{\perp} &= \frac{1}{\sqrt{Tn}}Z'M_Xu \\ &= \frac{1}{\sqrt{Tn}}Z'u-\left(\frac{1}{Tn}Z'X\right)\left(\frac{1}{Tn}X'X\right)^{-1}\left(\frac{1}{\sqrt{Tn}}X'u\right)\xrightarrow{d}\Psi_{Zu}-Q_{ZX}Q_{XX}^{-1}\Psi_{Xu}\end{aligned}$$

and

$$P_{Z^{\perp}}^{1/2}u^{\perp}=\left(\frac{1}{Tn}Z^{\perp\prime}Z^{\perp}\right)^{-1/2}\left(\frac{1}{\sqrt{Tn}}Z^{\perp\prime}u^{\perp}\right)\xrightarrow{d}\Omega^{-1/2}\left(\Psi_{Zu}-Q_{ZX}Q_{XX}^{-1}\Psi_{Xu}\right)=\sigma_{uu}^{1/2}z_u$$

since

$$\frac{1}{\sqrt{Tn}}X'u=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{n}}X_t'u_t\right)\xrightarrow{d}\Psi_{Xu},$$

$$\frac{1}{\sqrt{Tn}}Z'u=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{n}}Z_t'u_t\right)\xrightarrow{d}\Psi_{Zu}$$

and

$$\begin{aligned}\frac{1}{\sqrt{Tn}}Z^{\perp\prime}u^{\perp} &= \frac{1}{\sqrt{Tn}}Z'M_Xu \\ &= \frac{1}{\sqrt{Tn}}Z'u-\left(\frac{1}{Tn}Z'X\right)\left(\frac{1}{Tn}X'X\right)^{-1}\left(\frac{1}{\sqrt{Tn}}X'u\right)\xrightarrow{d}\Psi_{Zu}-Q_{ZX}Q_{XX}^{-1}\Psi_{Xu}\end{aligned}$$

as $(n,T) \overset{seq}{\to} \infty$.

Consider (4). Similarly,

$$\begin{aligned}
\frac{1}{T\sqrt{n}} Z^{\perp\prime} Y^{\perp} &= \frac{1}{T\sqrt{n}} Z' M_X Y \\
&= \frac{1}{T\sqrt{n}} Z'Y - \left(\frac{1}{Tn} Z'X\right) \left(\frac{1}{Tn} X'X\right)^{-1} \left(\frac{1}{T\sqrt{n}} X'Y\right) \overset{p}{\longrightarrow} Q_{ZZ}C - Q_{ZX}Q_{XX}^{-1}Q_{XZ}C = \Omega C
\end{aligned}$$

and

$$\frac{1}{\sqrt{T}} P_{Z^{\perp}}^{1/2} Y^{\perp} = \left(\frac{1}{Tn} Z^{\perp\prime} Z^{\perp}\right)^{-1/2} \left(\frac{1}{T\sqrt{n}} Z^{\perp\prime} Y^{\perp}\right) \overset{p}{\longrightarrow} \Omega^{-1/2}\Omega C = \Omega^{1/2} C = \lambda \Sigma_{VV}^{1/2}$$

using

$$\frac{1}{\sqrt{n}} Z_t' Y_t = \frac{1}{\sqrt{n}} Z_t' Z_t \Pi + \frac{1}{\sqrt{n}} Z_t' V_t = \left(\frac{1}{n} Z_t' Z_t\right) C + \frac{1}{\sqrt{n}} Z_t' V_t \overset{d}{\longrightarrow} Q_{ZZ}C + \Psi_{ZV},$$

$$\frac{1}{T\sqrt{n}} Z'Y = \frac{1}{T} \sum_{t=1}^{T} \frac{1}{\sqrt{n}} Z_t' Y_t = \frac{1}{T} \sum_{t=1}^{T} (Q_{ZZ}C + \Psi_{ZV}) \overset{p}{\longrightarrow} Q_{ZZ}C,$$

$$\frac{1}{\sqrt{n}} X_t' Y_t = \frac{1}{\sqrt{n}} X_t' Z_t \Pi + \frac{1}{\sqrt{n}} X_t' V_t = \left(\frac{1}{n} X_t' Z_t\right) C + \frac{1}{\sqrt{n}} X_t' V_t \overset{d}{\longrightarrow} Q_{XZ}C + \Psi_{XV}$$

and

$$\frac{1}{T\sqrt{n}} X'Y = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{1}{\sqrt{n}} X_t' Y_t\right) = \frac{1}{T} \sum_{t=1}^{T} (Q_{XZ}C + \Psi_{XV}) \overset{p}{\longrightarrow} Q_{XZ}C.$$

Using Lemmas (3) and (4), it is easy to see

$$\begin{aligned}
u^{\perp\prime} P_{Z^{\perp}} u^{\perp} &= \left(u^{\perp\prime} P_{Z^{\perp}}^{1/2}\right) \left(P_{Z^{\perp}}^{1/2} u^{\perp}\right) \overset{d}{\longrightarrow} \sigma_{uu} z_u' z_u, \\
\frac{1}{\sqrt{T}} Y^{\perp\prime} P_{Z^{\perp}} u^{\perp} &= \left(\frac{1}{\sqrt{T}} Y^{\perp\prime} P_{Z^{\perp}}^{1/2}\right) \left(P_{Z^{\perp}}^{1/2} u^{\perp}\right) \overset{d}{\longrightarrow} \sigma_{uu}^{1/2} \Sigma_{VV}^{1/2\prime} \lambda' z_u, \\
\frac{1}{T} Y^{\perp\prime} P_{Z^{\perp}} u^{\perp} &= \left(\frac{1}{\sqrt{T}} Y^{\perp\prime} P_{Z^{\perp}}^{1/2}\right) \left(\frac{1}{\sqrt{T}} P_{Z^{\perp}}^{1/2} u^{\perp}\right) \overset{p}{\longrightarrow} 0
\end{aligned}$$

and

$$\frac{1}{T} Y^{\perp\prime} P_{Z^{\perp}} Y^{\perp} = \left(\frac{1}{\sqrt{T}} Y^{\perp\prime} P_{Z^{\perp}}^{1/2}\right) \left(\frac{1}{\sqrt{T}} P_{Z^{\perp}}^{1/2} Y^{\perp}\right) \overset{p}{\longrightarrow} \Sigma_{VV}^{1/2\prime} \lambda' \lambda \Sigma_{VV}^{1/2}.$$

This proves (5). ∎

B Proof of Theorem 2

Proof. Let $\kappa_{Tn} = \sqrt{T} n (k-1)$. For (1), we write

$$\widehat{\beta}(k) = \beta + \left[Y^{\perp\prime} (I - kM_{Z^{\perp}}) Y^{\perp}\right]^{-1} \left[Y^{\perp\prime} (I - kM_{Z^{\perp}}) u^{\perp}\right].$$

Using $k=\left(1+\frac{\kappa_{Tn}}{\sqrt{T}n}\right)$ and $M_{Z^\perp}=I-P_{Z^\perp}$, by Lemmas (1), (2) and (5), we have

$$
\begin{aligned}
\frac{1}{T}Y^{\perp\prime}\left(I-kM_{Z^\perp}\right)Y^\perp &= \frac{1}{T}Y^{\perp\prime}\left[I-\left(1+\frac{\kappa_{Tn}}{\sqrt{T}n}\right)\left(I-P_{Z^\perp}\right)\right]Y^\perp \\
&= \left(1+\frac{\kappa_{Tn}}{\sqrt{T}n}\right)\left(\frac{1}{T}Y^{\perp\prime}P_{Z^\perp}Y^\perp\right)-\frac{\kappa_{Tn}}{\sqrt{T}}\left(\frac{1}{Tn}Y^{\perp\prime}Y^\perp\right)\xrightarrow{p}\Sigma_{VV}^{1/2\prime}\lambda^\prime\lambda\Sigma_{VV}^{1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{T}Y^{\perp\prime}\left(I-kM_{Z^\perp}\right)u^\perp &= \frac{1}{T}Y^{\perp\prime}\left[I-\left(1+\frac{\kappa_{Tn}}{\sqrt{T}n}\right)\left(I-P_{Z^\perp}\right)\right]u^\perp \\
&= \left(1+\frac{\kappa_{Tn}}{\sqrt{T}n}\right)\left(\frac{1}{T}Y^{\perp\prime}P_{Z^\perp}u^\perp\right)-\frac{\kappa_{Tn}}{\sqrt{T}}\left(\frac{1}{Tn}Y^{\perp\prime}u^\perp\right)\xrightarrow{p}0.
\end{aligned}
$$

Therefore,

$$
\widehat{\beta}\left(k\right)=\beta+\left[\frac{1}{T}Y^{\perp\prime}\left(I-kM_{Z^\perp}\right)Y^\perp\right]^{-1}\left[\frac{1}{T}Y^{\perp\prime}\left(I-kM_{Z^\perp}\right)u^\perp\right]\xrightarrow{p}\beta
$$

as $(n,T)\overset{seq}{\to}\infty$ joint with $\kappa_{Tn}\xrightarrow{d}\kappa=O_p\left(1\right)$.

Consider (2). By Lemmas (1), (2) and (5), we have

$$
\begin{aligned}
\frac{1}{\sqrt{T}}Y^{\perp\prime}\left(I-kM_{Z^\perp}\right)u^\perp &= \frac{1}{\sqrt{T}}Y^{\perp\prime}\left[I-\left(1+\frac{\kappa_{Tn}}{\sqrt{T}n}\right)\left(I-P_{Z^\perp}\right)\right]u^\perp \\
&= \left(1+\frac{\kappa_{Tn}}{\sqrt{T}n}\right)\left(\frac{1}{\sqrt{T}}Y^{\perp\prime}P_{Z^\perp}u^\perp\right)-\kappa_{Tn}\left(\frac{1}{Tn}Y^{\perp\prime}u^\perp\right) \\
&\quad \xrightarrow{d}\sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}\lambda^\prime z_u-\kappa\Sigma_{Vu}
\end{aligned}
$$

uniformly in κ. Therefore,

$$
\begin{aligned}
&\sqrt{T}\left[\widehat{\beta}\left(k\right)-\beta\right]+\left[\frac{1}{T}Y^{\perp\prime}\left(I-kM_{Z^\perp}\right)Y^\perp\right]^{-1}\kappa_{Tn}\left(\frac{1}{Tn}Y^{\perp\prime}u^\perp\right) \\
=&\left[\frac{1}{T}Y^{\perp\prime}\left(I-kM_{Z^\perp}\right)Y^\perp\right]^{-1}\left[\frac{1}{\sqrt{T}}Y^{\perp\prime}\left(I-kM_{Z^\perp}\right)u^\perp+\kappa_{Tn}\left(\frac{1}{Tn}Y^{\perp\prime}u^\perp\right)\right] \\
&\xrightarrow{d}\left(\Sigma_{VV}^{1/2\prime}\lambda^\prime\lambda\Sigma_{VV}^{1/2}\right)^{-1}\left(\sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}\lambda^\prime z_u\right)\sim N\left(0,\sigma_{uu}\left(\Sigma_{VV}^{1/2\prime}\lambda^\prime\lambda\Sigma_{VV}^{1/2}\right)^{-1}\right)
\end{aligned}
$$

as $(n,T)\overset{seq}{\to}\infty$ joint with $\kappa_{Tn}\xrightarrow{d}\kappa=O_p\left(1\right)$. Theorem 2 is proved if $\frac{1}{Tn}Y^{\perp\prime}u^\perp$ is replaced by Σ_{Vu}. ■