

Panel regression with random noise

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Abstract

The paper explores the effect of measurement errors on the estimation of a linear panel data model. The conventional fixed effects estimator, which ignores measurement errors, is biased. By correcting for the bias one can construct consistent and asymptotically normal estimators. In addition, we find estimates for the asymptotic variances of these estimators. The paper focuses on multiplicative errors, which are often deliberately added to the data in order to minimize their disclosure risk. They can be analyzed in a similar way as additive errors, but with some important and consequential differences.

Keywords: Panel regression, multiplicative measurement errors, bias correction, asymptotic variance, disclosure control.

1 Introduction

The paper explores the effect of measurement errors on the estimation of a linear panel data regression model. The conventional fixed effects least squares estimator, which ignores measurement errors, is biased. By correcting for the bias we can construct consistent and asymptotically normal estimators, where asymptotically here means that the number of sample units tends to infinity.

Measurement errors can be additive or multiplicative. Additive measurement errors in panel data models have been extensively studied in the literature, Griliches and Hausman (1986), Hsiao and Tay-

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lor (1991), Wansbeek and Koning (1991), Biørn (1996), Wansbeek (2001), Biørn and Krishnakumar (2008). But multiplicative measurement errors, though not uncommon in other models, have not found much attention in the context of panel data models.

The present paper is a first attempt to fill this gap. It was motivated by the various worldwide endeavors to find methods for masking data so that their disclosure risk becomes negligible, see, e.g., Domingo-Ferrer and Saygin (2008). Data and in particular panel data that are released to the public should be not only nominally but also factually anonymous. Making them anonymous in this sense can be done by (slightly) distorting them. The distortion, of course, should be such that the disturbing effects on any subsequent scientific analysis of the data should be minimal or should be amenable to correction. One way of perturbing data is to mix them with random noise, see Kim (1986) as an early reference. This can be done by adding random measurement errors to the data or by multiplying them with measurement errors. The latter procedure is often preferred, as it takes automatically into account that large values of a sensitive variable are more prone to disclosure and hence need to be better protected. In contrast to an additive error, a multiplicative error will distort large values more than small values.

Another aspect of statistical disclosure control techniques is that the procedure used is typically made known to the public. In our case, this means that the measurement error variance is known to the statistician working with the data.

Although linear panel regressions can also be estimated without this knowledge, we here assume that the error variance is known. This assumption not only leads to simpler estimators but also to more efficient ones. Indeed, we use this knowledge as prior information to construct consistent estimators. In addition, we find estimates for the asymptotic variances of these estimators.

We only deal with one type of estimators, the familiar “within” LS estimator and its corrections. It uses the “within” variances and covariances for each sample unit over time instead of the overall (total) variances and covariances. In doing so, the unobserved heterogeneity which is present in the panel data is eliminated. There are other estimators that can do the same, especially instrumental variable estimators that use lagged values of the variables as instruments. But in order for them to function properly some assumptions about the variables must be satisfied. E.g., for instrumental variables to work the variables must be autocorrelated. No such assumptions are needed for the “within” estimators.

Although this paper is mainly concerned with multiplicative mea-

surement errors, we also deal briefly with the additive case, just to show the parallel development in both cases.

While iid measurement errors are the main subject of our investigation, we also explore a special case of autocorrelated errors, a case which has been suggested especially for masking panel data: In addition to an iid part, the measurement error has a component which is random over the sample units but constant in time. It thus has a common factor structure, see Biørn(1996) and Hühne (2008).

The principles involved for constructing consistent estimators can be most easily presented in the context of a simple linear model with only one slope parameter. The procedures developed can then be extended to the case of a multiple regression, which, however, is not done in the present paper.

In Section 2 the linear panel model with measurement errors is presented. Section 3 introduces the within LS (naive) estimator of the slope parameter. In Section 4 the bias of the naive estimator is derived, and in Section 5 a corrected estimator is constructed. Section 6 deals with the case of a common factor structure in the error process. In Section 7 asymptotic variances for the various corrected estimators are presented. Section 8 has a simulation study, where the asymptotic properties of the estimators are studied under small to medium size samples. Section 9 concludes.

2 The model

2.1 Linear panel regression model

A panel consists of a sample of $i = 1, \dots, N$ sample units (individuals, households, companies) observed over $t = 1, \dots, T$ points of time (waves). Typically T is small and N is large. Asymptotics in this context therefore means asymptotics for $N \rightarrow \infty$.

For each individual and each wave, two variables are observed, a regressor (or independent) variable x and a response (or dependent) variable y . The panel data consists of the set (x_{it}, y_{it}) , $i = 1, \dots, N$, $t = 1, \dots, T$. We assume a linear relation between x and y as follows:

$$y_{it} = \gamma_i + \beta x_{it} + \varepsilon_{it}, \quad i = 1, \dots, N, \quad t = 1, \dots, T, \quad (1)$$

where ε is the error in the equation, and γ_i is the individual effect, giving rise to unobserved heterogeneity. β is the slope parameter to be estimated.

All variables including the γ_i are assumed to be random. However, the subsequent results, which are all of an asymptotic nature, remain

valid if instead the γ_i or x_{it} are taken to be deterministic as long as their empirical moments behave asymptotically (i.e., for $N \rightarrow \infty$) in the same way as if they were random. (Exact conditions could be formulated, but are not presented in this paper.)

The ε_{it} follow the usual assumptions: they are iid with mean 0 and variance σ_ε^2 and they are independent of the γ_i and x_{it} . As to the latter, we assume that the vectors $(\gamma_i, x_{i1}, \dots, x_{iT})$, $i = 1, \dots, N$, are iid. We make no assumptions about the joint distribution of $(\gamma_i, x_{i1}, \dots, x_{iT})$ except that the moments as far as necessary exist. In particular, the x_{it} may be autocorrelated and the γ_i may be correlated with the x_{it} .

2.2 Measurement errors

The variables x and y are not known to the statistician. Instead surrogate variables x^a and y^a , which are the original variables intermingled with error, are released to him. In the case of *additive errors* these are given by

$$x_{it}^a = x_{it} + u_{it}, \quad y_{it}^a = y_{it} + v_{it} \quad (2)$$

and for *multiplicative errors* by

$$x_{it}^a = x_{it}U_{it}, \quad y_{it}^a = y_{it}V_{it} \quad (3)$$

with

$$U_{it} = 1 + u_{it}, \quad V_{it} = 1 + v_{it}.$$

In both cases, the pairs (u_{it}, v_{it}) are iid with mean $(0,0)$ and variances σ_u^2 , σ_v^2 and covariance σ_{uv} . They are independent of all the (x_{it}, y_{it}) . If the measurement errors u and v are used to mask the data, they will typically be uncorrelated. However, to cover the general case, we do not assume $\sigma_{uv} = 0$.

3 Estimators

The conventional “within” estimator of β constructed from the true data is given by:

$$\hat{\beta} = \frac{S_{xy}}{S_{xx}}, \quad (4)$$

where

$$S_{xy} := \frac{1}{NT} \sum_i \sum_t (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i)$$

$$S_{xx} := \frac{1}{NT} \sum_i \sum_t (x_{it} - \bar{x}_i)^2$$

with $\bar{x}_i = \frac{1}{T} \sum_t x_{it}$ and $\bar{y}_i = \frac{1}{T} \sum_t y_{it}$. The use of the within variance S_{xx} and within covariance S_{xy} results in the elimination of the individual effects γ_i . S_{xx} and S_{xy} can also be written as

$$\begin{aligned} S_{xy} &= \overline{s_{xy}} \\ S_{xx} &= \overline{s_{xx}}, \end{aligned}$$

where

$$\begin{aligned} s_{xy} &:= (s_{xy})_i = \frac{1}{T} \sum_t (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) \\ s_{xx} &:= (s_{xx})_i = \frac{1}{T} \sum_t (x_{it} - \bar{x}_i)^2 \end{aligned}$$

are, respectively, the covariance of x and y and the variance of x for an individual i over time t , and the bar denotes averaging over $i = 1, \dots, N$, e.g.: $\overline{s_{xy}} = \frac{1}{N} \sum_i (s_{xy})_i$.

This estimator of β is not feasible, as it uses the true data, which are unknown to the statistician. A feasible estimator can be constructed in the same way but with the randomly perturbed data in place of the original data:

$$\hat{\beta}^a = \frac{S_{xy}^a}{S_{xx}^a} \quad (5)$$

with

$$\begin{aligned} S_{xy}^a &:= \frac{1}{NT} \sum_i \sum_t (x_{it}^a - \bar{x}_i^a)(y_{it}^a - \bar{y}_i^a) = \overline{s_{xy}^a} \\ S_{xx}^a &:= \frac{1}{NT} \sum_i \sum_t (x_{it}^a - \bar{x}_i^a)^2 = \overline{s_{xx}^a}, \end{aligned}$$

where the superscript a indicates the use of the variables x^a and y^a . This estimator may be called a naive estimator, as it does not take the perturbations into account. Indeed, the naive estimator turns out to be biased. Derivation of the bias is the subject of the next section.

4 Derivation of bias

4.1 Probability limits

In order to derive the bias of $\hat{\beta}^a$, we need to compute the probability limits of S_{xy}^a and S_{xx}^a . We do this separately for the two cases of additive and multiplicative errors.

4.1.1 Additive errors

From the definition of S_{xy}^a and S_{xy} and because of (2) it follows that

$$\begin{aligned} S_{xy}^a - S_{xy} &= \overline{s_{xy}^a} - \overline{s_{xy}} \\ &= \overline{s_{xv}} + \overline{s_{yu}} + \overline{s_{uv}}, \end{aligned} \quad (6)$$

where the last three terms are defined in the same way as $\overline{s_{xy}}$ above. As $N \rightarrow \infty$ these terms go to the corresponding expected values:

$$\text{plim}(S_{xy}^a - S_{xy}) = \mathbb{E}s_{xv} + \mathbb{E}s_{yu} + \mathbb{E}s_{uv}. \quad (7)$$

Note that, e.g., $\mathbb{E}s_{xv}$ is actually $\mathbb{E}(s_{xv})_i$, where $(s_{xv})_i = \frac{1}{T} \sum_t (x_{it} - \bar{x}_i)(v_{it} - \bar{v}_i)$. But since the expectation of this term is independent of i , we omit the index i . Now, because of the independence assumption, $\mathbb{E}s_{xv} = \mathbb{E}s_{yu} = 0$. As to $\mathbb{E}s_{uv}$, we have

$$\mathbb{E}s_{uv} = \left(1 - \frac{1}{T}\right) \sigma_{uv}.$$

Consequently,

$$\text{plim}(S_{xy}^a - S_{xy}) = \left(1 - \frac{1}{T}\right) \sigma_{uv}. \quad (8)$$

In a similar way or just by setting $y = x$ and $v = u$, we obtain

$$\text{plim}(S_{xx}^a - S_{xx}) = \left(1 - \frac{1}{T}\right) \sigma_u^2. \quad (9)$$

The factor $1 - \frac{1}{T} = \frac{T-1}{T}$, which appears in (8) and (9) and in many of the subsequent expressions is of course the correction for degrees of freedom. As typically T is not large in panel data, this factor can never be suppressed. It is an important distinction of panel data analysis from cross section data analysis.

4.1.2 Multiplicative errors

For multiplicative errors, we replace, in the arguments above, u_{it} and v_{it} with $x_{it}u_{it}$ and $y_{it}v_{it}$, respectively. We then obtain in place of (7)

$$\text{plim}(S_{xy}^a - S_{xy}) = \mathbb{E}s_{x(yv)} + \mathbb{E}s_{y(xu)} + \mathbb{E}s_{(xu)(yv)}, \quad (10)$$

where, e.g.,

$$s_{(xu)(yv)} := (s_{(xu)(yv)})_i = \frac{1}{T} \sum_t \left(x_{it}u_{it} - \frac{1}{T} \sum_t x_{it}u_{it} \right) \left(y_{it}v_{it} - \frac{1}{T} \sum_t y_{it}v_{it} \right).$$

Again by the independence assumption, $\mathbb{E}s_{x(yv)} = \mathbb{E}s_{y(xu)} = 0$. As to $\mathbb{E}s_{(xu)(yv)}$, we obtain

$$\begin{aligned}\mathbb{E}s_{(xu)(yv)} &= \frac{1}{T} \sum_t \mathbb{E}(x_{it}y_{it}u_{it}v_{it}) - \frac{1}{T^2} \sum_t \sum_s \mathbb{E}(x_{it}y_{is}u_{it}v_{is}) \\ &= \sigma_{uv} \mathbb{E}\bar{xy} - \frac{1}{T} \sigma_{uv} \mathbb{E}\bar{xy} \\ &= \left(1 - \frac{1}{T}\right) \sigma_{uv} \mathbb{E}\bar{xy},\end{aligned}\tag{11}$$

where $\bar{xy} := (\bar{xy})_i = \frac{1}{T} \sum_t (x_{it}y_{it})$. Consequently,

$$\text{plim}(S_{xy}^a - S_{xy}) = \left(1 - \frac{1}{T}\right) \sigma_{uv} \mathbb{E}\bar{xy}.\tag{12}$$

By setting $y = x$ and $v = u$, we finally obtain

$$\text{plim}(S_{xx}^a - S_{xx}) = \left(1 - \frac{1}{T}\right) \sigma_u^2 \mathbb{E}\bar{x}^2.\tag{13}$$

4.2 Bias

Substituting the probability limits in (5), we immediately obtain expressions for the probability limit of the naive estimator and thus implicitly for the (asymptotic) bias.

4.2.1 Additive errors

By (5), (8), and (9),

$$\begin{aligned}\text{plim}\hat{\beta}^a &= \frac{\text{plim}S_{xy} + \left(1 - \frac{1}{T}\right) \sigma_{uv}}{\text{plim}S_{xx} + \left(1 - \frac{1}{T}\right) \sigma_u^2} \\ &= \frac{\beta \mathbb{E}s_{xx} + \left(1 - \frac{1}{T}\right) \sigma_{uv}}{\mathbb{E}s_{xx} + \left(1 - \frac{1}{T}\right) \sigma_u^2},\end{aligned}\tag{14}$$

where we used $\text{plim}S_{xx} = \mathbb{E}s_{xx}$ and $\text{plim}S_{xy} = \beta \mathbb{E}s_{xx}$, which follows from (1).

4.2.2 Multiplicative errors

By (5), (12), (13), and the model equation (1),

$$\begin{aligned}\text{plim}\hat{\beta}^a &= \frac{\text{plim}S_{xy} + \left(1 - \frac{1}{T}\right) \sigma_{uv} \mathbb{E}\bar{xy}}{\text{plim}S_{xx} + \left(1 - \frac{1}{T}\right) \sigma_u^2 \mathbb{E}\bar{x}^2} \\ &= \frac{\beta \mathbb{E}s_{xx} + \left(1 - \frac{1}{T}\right) \sigma_{uv} \mathbb{E}\bar{xy}}{\mathbb{E}s_{xx} + \left(1 - \frac{1}{T}\right) \sigma_u^2 \mathbb{E}\bar{x}^2}.\end{aligned}\tag{15}$$

In the additive case, the bias depends on $\mathbb{E}s_{xx}$, and in the multiplicative case, it depends in addition on $\mathbb{E}\bar{x}^2$ and, if $\sigma_{uv} \neq 0$, on $\mathbb{E}\bar{xy}$. These expectations in turn depend on the joint distribution of the vector $(\gamma_i, x_{i1}, \dots, x_{iT})$, e.g., on whether the x_{it} are iid or are autocorrelated or whether the individual effects are uncorrelated or correlated with the x_{it} . To derive more explicit expressions for the bias, one would thus have to know the stochastic law governing the γ_i and the x_{it} . We do not pursue this line, but see Ronning (2007). It turns out that for constructing consistent estimators it is not necessary to know the distribution of $(\gamma_i, x_{i1}, \dots, x_{iT})$. However, the asymptotic variances of the estimators depend on this distribution, see Section 8.

5 Bias correction

We can derive correction formulas either by solving the bias formulas (14) and (15), respectively, for β or by using (4) directly. Taking the latter course, we construct a corrected estimator of β as

$$\hat{\beta}^c = \frac{\hat{S}_{xy}}{\hat{S}_{xx}}, \quad (16)$$

where \hat{S}_{xx} and \hat{S}_{xy} are suitable “estimates” of S_{xx} and S_{xy} , respectively, which we obtain from Section 4.1 for the additive as well as for the multiplicative case.

5.1 Additive errors

In the additive case, we use (8) and (9) to estimate S_{xy} and S_{xx} , respectively, and obtain the following corrected estimator of β :

$$\hat{\beta}^c = \frac{S_{xy}^a - (1 - \frac{1}{T})\sigma_{uv}}{S_{xx}^a - (1 - \frac{1}{T})\sigma_u^2}. \quad (17)$$

5.2 Multiplicative errors

In the multiplicative case, we use (12) and (13) to estimate S_{xy} and S_{xx} , respectively, and obtain the following corrected estimator of β :

$$\hat{\beta}^c = \frac{S_{xy}^a - (1 - \frac{1}{T})\sigma_{uv}\hat{\mathbb{E}}\bar{xy}}{S_{xx}^a - (1 - \frac{1}{T})\sigma_u^2\hat{\mathbb{E}}\bar{x}^2}, \quad (18)$$

where $\hat{\mathbb{E}}\overline{xy}$ and $\hat{\mathbb{E}}\overline{x^2}$ are suitable estimates of $\mathbb{E}\overline{xy}$ and $\mathbb{E}\overline{x^2}$. In order to find these estimates, we consider

$$\overline{\overline{x^a y^a}} := \frac{1}{NT} \sum_i \sum_t x_{it}^a y_{it}^a. \quad (19)$$

We have

$$\overline{\overline{x^a y^a}} = \frac{1}{N} \sum_i \frac{1}{T} \sum_t x_{it} y_{it} U_{it} V_{it},$$

which tends in probability to

$$\frac{1}{T} \sum_t \mathbb{E}(x_{it} y_{it} U_{it} V_{it}) = \mathbb{E} \left(\frac{1}{T} \sum_t x_{it} y_{it} \right) \mathbb{E}(UV) = \mathbb{E}\overline{xy} (\sigma_{uv} + 1).$$

It follows that

$$\hat{\mathbb{E}}\overline{xy} := \frac{\overline{\overline{x^a y^a}}}{1 + \sigma_{uv}} \quad (20)$$

is a consistent estimate of $\mathbb{E}\overline{xy}$. Similarly,

$$\hat{\mathbb{E}}\overline{x^2} := \frac{\overline{\overline{x^{a2}}}}{1 + \sigma_u^2} \quad (21)$$

is a consistent estimate of $\mathbb{E}\overline{x^2}$, where

$$\overline{\overline{x^{a2}}} := \frac{1}{NT} \sum_i \sum_t x_{it}^{a2}. \quad (22)$$

With the abbreviations $q_{uv} := (1 - \frac{1}{T}) \frac{\sigma_{uv}}{1 + \sigma_{uv}}$ and $q_u := (1 - \frac{1}{T}) \frac{\sigma_u^2}{1 + \sigma_u^2}$, we can write the corrected estimator as

$$\hat{\beta}^c = \frac{S_{xy}^a - q_{uv} \overline{\overline{x^a y^a}}}{S_{xx}^a - q_u \overline{\overline{x^{a2}}}}. \quad (23)$$

6 Common factor in the error process

6.1 The model

Höhne (2008) suggests a different kind of (additive or multiplicative) random noise. It is characterized by having a common factor structure:

$$\begin{aligned} u_{it} &= d_i + u_{it}^* \\ v_{it} &= e_i + v_{it}^*, \end{aligned} \quad (24)$$

where d_i and e_i are random variables with $\mathbb{E}d_i = \mathbb{E}e_i = 0$ and finite variances.

In studying disclosure control techniques, Höhne assumes a very special distribution for these variables, e.g., d_i is of the form $d_i = \delta D_i$, where D_i is a variable which takes the values 1 and -1 , each with probability $\frac{1}{2}$, and δ is a known constant. Moreover, he sets $d_i = e_i$, which is motivated by the idea that then ratios of variables such as x/y are only slightly affected by the random noise, see Ronning (2009). We do not make these special assumptions, but, of course, this case is covered by our general approach. However, we do require independence and iid assumptions for these new error terms similar to those of Section 2.2.

Independence assumptions: As before, the set of pairs (u_{it}, v_{it}) is independent of the set of pairs (x_{it}, y_{it}) . In addition, the set of pairs (e_i, d_i) is independent of the set of pairs (u_{it}^*, v_{it}^*) .

Iid assumptions: The pairs (u_{it}^*, v_{it}^*) and the pairs (d_i, e_i) are iid.

These assumptions differ from the corresponding ones in Section 2.2 in that the u_{it} and v_{it} are no more iid, but are serially correlated. Instead, the u_{it}^* and v_{it}^* are now iid. This has consequences for the derivation of bias and correction formulas.

As to the relation between d and e , there are two main cases considered in the literature. They can be independent or they can be equal.

There is also the possibility that we have iid pairs (d_{it}, e_{it}) instead of (d_i, e_i) . In this case, the previous theory goes through unchanged.

6.2 Additive errors

In the additive case, the within variances and covariances depend on the error terms u_{it} and v_{it} only through the differences $u_{it} - \bar{u}_i$ and $v_{it} - \bar{v}_i$, see (6). But since

$$\begin{aligned} u_{it} - \bar{u}_i &= u_{it}^* - \bar{u}_i^* \\ v_{it} - \bar{v}_i &= v_{it}^* - \bar{v}_i^*, \end{aligned}$$

we get the the same results as before except that u_{it} and v_{it} have to be replaced with u_{it}^* and v_{it}^* , respectively.

Thus the *bias* is given by

$$\text{plim} \hat{\beta}^a = \frac{\beta \mathbb{E} s_{xx} + (1 - \frac{1}{T}) \sigma_{u^*v^*}}{\mathbb{E} s_{xx} + (1 - \frac{1}{T}) \sigma_{u^*}^2},$$

and the *corrected* estimator of β is

$$\hat{\beta}^c = \frac{S_{xy}^a - (1 - \frac{1}{T}) \sigma_{u^*v^*}}{S_{xx}^a - (1 - \frac{1}{T}) \sigma_{u^*}^2}.$$

6.3 Multiplicative errors

In the multiplicative case, we need to proceed more carefully. We start with the probability limits of the within variances and covariances as in Section 4.1. The general formula (10) for the probability limit of S_{xy}^a is still true and, again because of the independence assumption, $\mathbb{E}s_{x(yv)} = \mathbb{E}s_{y(xu)} = 0$. However, $\mathbb{E}s_{(xu)(yv)}$ is different. First note that

$$\begin{aligned} x_{it}u_{it} - (\overline{xu})_i &= d_i(x_{it} - \bar{x}_i) + x_{it}u_{it}^* - (\overline{xu^*})_i \\ y_{it}v_{it} - (\overline{yv})_i &= e_i(y_{it} - \bar{y}_i) + y_{it}v_{it}^* - (\overline{yv^*})_i, \end{aligned}$$

where, e.g., $(\overline{xu})_i = \frac{1}{T} \sum_t x_{it}u_{it}$. It follows that

$$\begin{aligned} \mathbb{E}s_{(xu)(yv)} &= \mathbb{E} \frac{1}{T} \sum_t (x_{it}u_{it} - (\overline{xu})_i)(y_{it}v_{it} - (\overline{yv})_i) \\ &= \mathbb{E} \left[d_i e_i \frac{1}{T} \sum_t (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) \right. \\ &\quad + d_i \frac{1}{T} \sum_t (x_{it} - \bar{x}_i)(y_{it}v_{it}^* - (\overline{yv^*})_i) \\ &\quad + e_i \frac{1}{T} \sum_t (y_{it} - \bar{y}_i)(x_{it}u_{it}^* - (\overline{xu^*})_i) \\ &\quad \left. + \frac{1}{T} \sum_t (x_{it}u_{it}^* - (\overline{xu^*})_i)(y_{it}v_{it}^* - (\overline{yv^*})_i) \right] \\ &= \mathbb{E}(de)\mathbb{E}s_{xy} + \mathbb{E}s_{(xu^*)(yv^*)} \\ &= \sigma_{de}\mathbb{E}s_{xy} + \left(1 - \frac{1}{T}\right)\sigma_{u^*v^*}\mathbb{E}\overline{xy}. \end{aligned}$$

Therefore,

$$\text{plim}(S_{xy}^a - S_{xy}) = \sigma_{de}\mathbb{E}s_{xy} + \left(1 - \frac{1}{T}\right)\sigma_{u^*v^*}\mathbb{E}\overline{xy}. \quad (25)$$

Similarly with (x, u, d) in place of (y, v, e) ,

$$\text{plim}(S_{xx}^a - S_{xx}) = \sigma_d^2\mathbb{E}s_{xx} + \left(1 - \frac{1}{T}\right)\sigma_{u^*}^2\mathbb{E}\overline{x^2}. \quad (26)$$

(25) and (26) imply the following *bias* formula

$$\begin{aligned} \text{plim}\hat{\beta}^a &= \frac{\text{plim}S_{xy} + \sigma_{de}\mathbb{E}s_{xy} + \left(1 - \frac{1}{T}\right)\sigma_{u^*v^*}\mathbb{E}\overline{xy}}{\text{plim}S_{xx} + \sigma_d^2\mathbb{E}s_{xx} + \left(1 - \frac{1}{T}\right)\sigma_{u^*}^2\mathbb{E}\overline{x^2}} \\ &= \frac{\beta(1 + \sigma_{de})\mathbb{E}s_{xx} + \left(1 - \frac{1}{T}\right)\sigma_{u^*v^*}\mathbb{E}\overline{xy}}{(1 + \sigma_d^2)\mathbb{E}s_{xx} + \left(1 - \frac{1}{T}\right)\sigma_{u^*}^2\mathbb{E}\overline{x^2}}. \end{aligned} \quad (27)$$

A *bias correction* for estimating β is found by evaluating

$$\hat{\beta}^c = \frac{\hat{S}_{xy}}{\hat{S}_{xx}}. \quad (28)$$

We find an estimator for S_{xy} from (25) as follows:

$$\hat{S}_{xy} = S_{xy}^a - \hat{S}_{xy} \sigma_{de} - \left(1 - \frac{1}{T}\right) \sigma_{u^*v^*} \hat{\mathbb{E}}\overline{xy},$$

where $\hat{\mathbb{E}}\overline{xy}$ is a suitable estimator of $\mathbb{E}\overline{xy}$, and so

$$\hat{S}_{xy} = \frac{S_{xy}^a - \left(1 - \frac{1}{T}\right) \sigma_{u^*v^*} \hat{\mathbb{E}}\overline{xy}}{1 + \sigma_{de}}. \quad (29)$$

Similarly,

$$\hat{S}_{xx} = \frac{S_{xx}^a - \left(1 - \frac{1}{T}\right) \sigma_{u^*}^2 \hat{\mathbb{E}}\overline{x^2}}{1 + \sigma_d^2}. \quad (30)$$

Substituting (29) and (30) in (28) results in

$$\hat{\beta}^c = \frac{S_{xy}^a - \left(1 - \frac{1}{T}\right) \sigma_{u^*v^*} \hat{\mathbb{E}}\overline{xy}}{S_{xx}^a - \left(1 - \frac{1}{T}\right) \sigma_{u^*}^2 \hat{\mathbb{E}}\overline{x^2}} \cdot \frac{1 + \sigma_d^2}{1 + \sigma_{de}}. \quad (31)$$

It remains to construct estimators of $\mathbb{E}\overline{xy}$ and $\mathbb{E}\overline{x^2}$. We start with

$$\begin{aligned} \overline{x^a y^a} &:= \frac{1}{N} \frac{1}{T} \sum_i \sum_t x_{it}^a y_{it}^a \\ &= \frac{1}{N} \sum_i \frac{1}{T} \sum_t x_{it} y_{it} (1 + d_i + u_{it}^*) (1 + e_i + v_{it}^*) \\ &= \frac{1}{N} \sum_i \frac{1}{T} \sum_t x_{it} y_{it} (1 + d_i + e_i + d_i e_i + u_{it}^* + v_{it}^* + d_i v_{it}^* + e_i u_{it}^* + u_{it}^* v_{it}^*), \end{aligned}$$

which converges in probability to $\mathbb{E}\overline{xy}(1 + \sigma_{ed} + \sigma_{u^*v^*})$. An estimator of $\mathbb{E}\overline{xy}$ is thus given by

$$\hat{\mathbb{E}}\overline{xy} = \frac{\overline{x^a y^a}}{1 + \sigma_{ed} + \sigma_{u^*v^*}}. \quad (32)$$

Similarly,

$$\hat{\mathbb{E}}\overline{x^2} = \frac{\overline{x^{a2}}}{1 + \sigma_d^2 + \sigma_{u^*}^2}. \quad (33)$$

7 Asymptotic variances

7.1 Estimating equations

The asymptotic variances of the various estimators can be found from the general theory of unbiased estimating equations. Let $z_i, i = 1, \dots, N$, be a set of iid vector-valued random variables and θ a p -dimensional parameter vector pertaining to the distribution of z . Suppose a p -dimensional vector-valued function ψ of z and θ exists such that $\mathbb{E}\psi(z, \theta) = 0$. We call ψ an unbiased estimating function and the equation

$$\sum_{i=1}^N \psi(z_i, \hat{\theta}) = 0$$

an unbiased estimating equation, Heyde (1997). Under general conditions, the solution $\hat{\theta}$ to this equation exists uniquely, at least for large N , and is a consistent estimator of θ . In addition, $\hat{\theta}$ is asymptotically normal with an asymptotic covariance matrix given by the sandwich formula

$$\mathbb{V}(\hat{\theta}) = \frac{1}{N} (\mathbb{E}\psi_\theta)^{-1} \mathbb{E}\psi\psi^\top \mathbb{E}(\psi_\theta)^{-\top},$$

where

$$\psi_\theta := \frac{\partial \psi(z, \theta)}{\partial \theta^\top},$$

which is a $(p \times p)$ -matrix. A consistent estimator of the asymptotic covariance matrix is given by

$$\hat{\mathbb{V}}(\hat{\theta}) = \frac{1}{N} (\overline{\psi_\theta})^{-1} \overline{\psi\psi^\top} (\overline{\psi_\theta})^{-\top},$$

where

$$\overline{\psi_\theta} = \frac{1}{N} \sum_i \psi_\theta(z_i, \hat{\theta}) \tag{34}$$

$$\overline{\psi\psi^\top} = \frac{1}{N} \sum_i \psi(z_i, \hat{\theta})\psi(z_i, \hat{\theta})^\top = \frac{1}{N} \Psi^\top \Psi, \tag{35}$$

where Ψ^\top is the matrix $(\psi(z_1, \hat{\theta}), \dots, \psi(z_N, \hat{\theta}))$.

In the context of our panel model, $z_i = (x_{i1}, \dots, x_{iT}, y_{i1}, \dots, y_{iT})$ is the data vector for individual i of the panel population. The parameter vector θ has β in the first component as our parameter of interest and in addition possibly other (nuisance) parameters in the remaining components. The asymptotic variance of $\hat{\beta}$ is then given by

$$\hat{\mathbb{V}}(\hat{\beta}) = e_1^\top \hat{\mathbb{V}}(\hat{\theta}) e_1,$$

where $e_1 = (1, 0, \dots, 0)^\top$ is the p -dimensional first unit vector.

We apply this theory to the various error corrected estimators of our panel model by constructing an estimating function for each of the estimators.

7.2 Additive errors

From the corrected estimator of β (17), we obtain the estimating function

$$\psi = \psi(x, y, \beta) = (s_{xx}^a - (1 - \frac{1}{T})\sigma_u^2)\beta - (s_{xy}^a - (1 - \frac{1}{T})\sigma_{uv})$$

and

$$\psi_\beta = s_{xx}^a - (1 - \frac{1}{T})\sigma_u^2.$$

In this case, ψ is a scalar function, and so $\psi\psi^\top = \psi^2$.

7.3 Multiplicative errors

In the case of multiplicative errors, $\theta = (\beta, \mathbb{E}x^2, \mathbb{E}xy)^\top$, and the estimating function for θ , as implied by (18), (20), and (21), is given by a vector $\psi = (\psi_1, \psi_2, \psi_3)^\top$ with

$$\begin{aligned}\psi_1 &= [s_{xx}^a - (1 - \frac{1}{T})\sigma_u^2 \mathbb{E}x^2]\beta - [s_{xy}^a - (1 - \frac{1}{T})\sigma_{uv} \mathbb{E}xy] \\ \psi_2 &= (1 + \sigma_u^2)\mathbb{E}x^2 - x^{a2} \\ \psi_3 &= (1 + \sigma_{uv})\mathbb{E}xy - x^a y^a.\end{aligned}$$

For ψ_θ , we find

$$\psi_\theta = \begin{pmatrix} s_{xx}^a - (1 - \frac{1}{T})\sigma_u^2 \mathbb{E}x^2 & -(1 - \frac{1}{T})\sigma_u^2 \beta & (1 - \frac{1}{T})\sigma_{uv} \\ 0 & 1 + \sigma_u^2 & 0 \\ 0 & 0 & 1 + \sigma_{uv} \end{pmatrix}.$$

In case $\sigma_{uv} = 0$, ψ_3 and the last row and last column of ψ_θ can be dropped.

7.4 Common factor structure

If the measurement errors have a common factor structure, the estimators of Section 6 are relevant.

For *additive* errors, we have, according to Section 6.2, the same estimating function as in Subsection 7.2 except that σ_u^2 and σ_{uv} have to be replaced with $\sigma_{u^*}^2$ and $\sigma_{u^*v^*}$, respectively. Thus

$$\psi = \psi(x, y, \beta) = (s_{xx}^a - (1 - \frac{1}{T})\sigma_{u^*}^2)\beta - (s_{xy}^a - (1 - \frac{1}{T})\sigma_{u^*v^*}).$$

For *multiplicative* errors, we find the estimating function from (31). We have $\theta = (\beta, \mathbb{E}x^2, \mathbb{E}xy)^\top$ and $\psi = (\psi_1, \psi_2, \psi_3)^\top$ with

$$\begin{aligned}\psi_1 &= [s_{xx}^a - (1 - \frac{1}{T})\sigma_{u^*}^2 \mathbb{E}x^2] \beta - [s_{xy}^a - (1 - \frac{1}{T})\sigma_{u^*v^*} \mathbb{E}xy] \frac{1 + \sigma_d^2}{1 + \sigma_{de}} \\ \psi_2 &= (1 + \sigma_d^2 + \sigma_{u^*}^2) \mathbb{E}x^2 - \overline{x^2} \\ \psi_3 &= (1 + \sigma_{de} + \sigma_{u^*v^*}) \mathbb{E}xy - \overline{x^a y^a}.\end{aligned}$$

For ψ_θ , we find

$$\psi_\theta = \begin{pmatrix} s_{xx}^a - (1 - \frac{1}{T})\sigma_{u^*}^2 \mathbb{E}x^2 & -(1 - \frac{1}{T})\sigma_{u^*}^2 \beta & (1 - \frac{1}{T})\sigma_{u^*v^*} \frac{1 + \sigma_d^2}{1 + \sigma_{de}} \\ 0 & 1 + \sigma_d^2 + \sigma_{u^*}^2 & 0 \\ 0 & 0 & 1 + \sigma_{de} + \sigma_{u^*v^*} \end{pmatrix}.$$

Again, if $\sigma_{u^*v^*} = 0$, ψ_3 and the last row and last column of ψ_θ can be dropped.

It may be noted that in all the above cases the asymptotic variance of $\hat{\beta}$ can also be computed by the delta method. An example is given in the appendix.

8 Simulation

In our simulation study we analyze the performance of both the naive estimator (4) and the corrected estimators when both x and y are observed with multiplicative measurement errors. We do not present results for the more familiar case of additive errors.

For the iid case, as given in (3), the corrected estimator is given in (18) and for the (multiplicative) common factor model (24) the corrected estimator is given in (31). In all scenarios we used 2000 replications.

For the regressor variables we assumed the stationary AR(1) process

$$x_{it} = \phi + \rho x_{i,t-1} + \omega_{it} \quad (36)$$

with $|\rho| < 1$ and $\mathbb{E}\omega_{it} = 0, \mathbb{V}\omega_{it} = \sigma_\omega^2$ for all $i = 1, \dots, N$ and $t = 1, \dots, T$. We use

$$\phi = (1 - \rho)\mathbb{E}x_{it}, \sigma_\omega^2 = \mathbb{V}x_{it}(1 - \rho^2),$$

so that for each ρ (see below) we have the same expected value and variance of all x_{it} .

In order to study the effect of correlation of the individual effect γ with the regressor x we assume that the effect is given by

$$\gamma_i = [\bar{x}_i - \mathbb{E}\bar{x}_i] \lambda + w_i,$$

where w_i is normal white noise with expectation 0 and variance σ_w^2 distributed independently of x . The variance of w_i is defined by

$$\sigma_w^2 = \mathbb{V}\gamma_i - \lambda^2 \sigma_{\bar{x}}^2,$$

so that the variance of the individual component is kept constant for any λ (see below). This specification of correlated individual effects has been proposed by Biørn (1996) p. 260. Note that, for positive λ , the correlation of γ_i with \bar{x}_i is given by

$$\text{Corr}[\gamma_i, \bar{x}_i] = \frac{1}{\sqrt{1 + \frac{\sigma_w^2}{\lambda^2 \sigma_{\bar{x}}^2}}}.$$

This correlation will tend to 1 for $\lambda \rightarrow \infty$ and to 0 for $T \rightarrow \infty$.

When studying the common factor structure we use the special specification of Hühne(2008), which we already mentioned in section 6.1: We set $d_i = e_i$ in (24) and use the special structure

$$e_i = \delta D \quad \text{with} \quad D = \begin{cases} 1 & \text{with probability 0.5} \\ -1 & \text{with probability 0.5} \end{cases}$$

This special specification implies $\sigma_d^2 = \sigma_{de}$ so that the corrected estimator from (31) is given by

$$\hat{\beta}^c = \frac{S_{xy}^a - (1 - \frac{1}{T})\sigma_{u^*v^*} \hat{\mathbb{E}}\bar{x}\bar{y}}{S_{xx}^a - (1 - \frac{1}{T})\sigma_{u^*}^2 \hat{\mathbb{E}}\bar{x}^2}.$$

The following parameters were fixed throughout the whole simulation study: In the linear panel model (1) we used

$$\beta = 1, \mathbb{E}x_{it} = 2, \mathbb{V}x_{it} = 1.5^2, \mathbb{V}\varepsilon_{it} = 0.5^2, \mathbb{V}\gamma_i = 1,$$

and for the measurement errors we used

$$\mathbb{V}u_{it} = 0.2^2, \mathbb{V}v_{it} = 0.2^2.$$

for the iid case (3) and

$$\delta = 0.14, \mathbb{V}u_{it}^* = 0.14^2, \mathbb{V}v_{it}^* = 0.14^2$$

for the common factor case (3). Note that the total variance of measurement error in the latter case is given by $\mathbb{V}u_{it} = \delta^2 + \mathbb{V}u_{it}^* = 0.14^2 + 0.14^2 = 0.0392$ which is (almost) equal to the variance in case of (3).

For both the iid case and the common factor case, we studied the effects of varying the sample size N , the number of waves T , the

Table 1: Parameter variation in the simulation study

parameter	values used
N	100 ; 1000
T	3 ; 10
ϱ	-0.5 ; 0 ; +0.5
$\rho_{uv}(\rho_{u^*v^*})$	-0.9 ; 0 ; +0.9
λ	0 ; 1

autoregressive parameter ϱ and the correlation between u and v , which we denote by ρ_{uv} ($\rho_{u^*v^*}$ for the common factor model). Table 1 has the details. We use alternatively a "moderate" and a large sample size. The number of waves is kept very small. The autoregressive parameter specifies both positive and negative autocorrelation besides the case of no autocorrelation. Additionally, we consider zero and non-zero correlation between u and v (u^* and v^* for the common factor model). Finally, a non-zero parameter λ indicates correlation of individual effects with the regressor x .

The four tables contain the simulation results for the iid case (tables 2 and 3) and for the common factor case (tables 4 and 5). For each case, the second table reports results concerning correlation of individual effects with the regressor. In all four tables we use the following notation: $\hat{\beta}$ and $s_{\hat{\beta}}$ give mean and standard deviation of the 2000 replications concerning the naive estimator, $\hat{\beta}^c$ and $s_{\hat{\beta}^c}$ the corresponding results for the corrected estimator. $\hat{\sigma}_{\hat{\beta}^c}$ denotes the estimate of the theoretical (asymptotic) standard deviation discussed in section 7 and $s_{\hat{\sigma}_{\hat{\beta}^c}}$ reports the standard error of this estimate. Finally, $q_{\hat{\beta}^c}^\alpha$ is the α -quantile of the corrected estimator for three different levels of α .

Tables 2 , 3, 4 and 5 about here

For large samples ($N = 1000$), our simulations support our theoretical findings: The corrected slope estimator $\hat{\beta}^c$ shows practically no bias, and the average estimate of the theoretical (asymptotic) standard deviation of the estimator $\hat{\beta}^c$ corresponds very accurately to the empirical variance of the estimates $\hat{\beta}^c$ in the simulation runs. The asymptotic results seem to apply almost as well to samples of small to medium size ($N = 100$): the corrected slope estimator shows hardly any bias, and the theoretical standard deviation still corresponds rather closely to the empirical standard deviation. Of course, for smaller N , these standard deviations are (about three times) larger.

The simulations also highlight the considerable amount of bias in the uncorrected (naive) estimator of the slope parameter. The bias

tends to increase for increasing ρ and for decreasing ρ_{uv} – the latter in accordance with (15). The bias is considerably smaller for errors with a common factor structure, which is plausible considering the fact that, by using inner variances and covariances for constructing the estimator, the common factor is largely eliminated – it is completely eliminated in the additive case – so that the, much smaller, remaining error components u^* and v^* are now relevant for the bias. The presence of a correlation between individual effect and regressor has almost no effect on the bias. If at all, the bias is only a tiny bit larger in the case of correlation.

The standard deviation of the corrected estimator can also be seen to depend on the various model parameters. It decreases for increasing T , decreasing ρ , and increasing ρ_{uv} . It is a good deal smaller in the common factor case. The dependence on λ is negligible. The standard deviation of the corrected estimator is, of course, larger than for the naive estimator, but not very much. The increase in variance is outweighed by the elimination of bias. Finally, it may be noted that the estimate of the standard deviation is very precise in view of its own standard deviation, in particular for large N .

9 Conclusion

Measurement errors in a linear regression result in biased estimates of the slope parameter when Least Squares is applied without regard to the measurement errors. This is true for panel data models just as for cross sectional models, except that, in panel models, within LS estimators instead of the ordinary LS estimators are used in order to get rid of the unobservable individual effects. As a result of this difference, a degree of freedoms factor enters the bias formula, which is not present in the well-known bias formula for cross sectional data. When correcting for the bias in panel data models, this factor has again to be taken into account.

We focus our investigation on multiplicative iid. errors. They can be treated in a similar way as the more conventional additive errors, but with some characteristic differences. In the bias formula as well as in the expression for the bias corrected estimator, nuisance parameters appear, which have to be estimated, too. Their presence results in a substantially more complicated computation of the asymptotic variance of the slope estimator than in the additive case. The variance is computed with the help of the sandwich formula, which, however, has to take the nuisance parameters into account.

The results for iid errors can be generalized to the case of a common factor structure in the error process. Again, bias corrected estimators

and asymptotic variances can be derived both for the additive and for the multiplicative case.

An extensive simulation study was carried out. It fully corroborates our theoretical findings on the asymptotics of our estimators and shows that the asymptotic results seem to apply almost as well to samples of small to medium size ($N = 100$). The simulations also make evident the dependence of the asymptotic variance on the various model parameters, e.g., on the autocorrelation of the regressor variable or on the correlation of regressor variable and individual effect variable.

Finally, they show how close the asymptotic variance of the corrected estimator may come to that of the uncorrected estimator, at least for large N . Thus the correction is fully justified both on the ground that it eliminates the bias and that it implies only a small increase in variance.

10 Literature

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A Example for the delta method

Consider the estimator (18) of β for the special case $\sigma_{uv} = 0$:

$$\hat{\beta}^c = \frac{\overline{s_{xy}^a}}{\overline{s_{xx}^a} - q\overline{x^{a2}}}, \quad q := \frac{T-1}{T} \frac{\sigma_u^2}{1 + \sigma_u^2}.$$

By the delta method, an estimate of the asymptotic variance of $\hat{\beta}^c$ is given by

$$\hat{\mathbb{V}}\hat{\beta}^c = \frac{\hat{\beta}^{c2}}{S_{xy}^{a2}} \begin{pmatrix} 1 & -\hat{\beta}^c & q\hat{\beta}^c \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} \begin{pmatrix} 1 \\ -\hat{\beta}^c \\ q\hat{\beta}^c \end{pmatrix},$$

where

$$\begin{aligned} v_{11} &= \hat{\mathbb{V}}(\overline{s_{xy}^a}) = \frac{1}{N} (\overline{s_{xy}^{a2}} - \overline{s_{xy}^a}^2) \\ v_{12} &= \hat{\mathbb{C}ov}(\overline{s_{xy}^a}, \overline{s_{xx}^a}) = \frac{1}{N} (\overline{s_{xy}^a s_{xx}^a} - \overline{s_{xy}^a} \overline{s_{xx}^a}) \\ v_{13} &= \hat{\mathbb{C}ov}(\overline{s_{xy}^a}, \overline{x^{a2}}) = \frac{1}{N} (\overline{s_{xy}^a x^{a2}} - \overline{s_{xy}^a} \overline{x^{a2}}) \\ v_{22} &= \hat{\mathbb{V}}(\overline{s_{xx}^a}) = \frac{1}{N} (\overline{s_{xx}^{a2}} - \overline{s_{xx}^a}^2) \\ v_{23} &= \hat{\mathbb{C}ov}(\overline{s_{xx}^a}, \overline{x^{a2}}) = \frac{1}{N} (\overline{s_{xx}^a x^{a2}} - \overline{s_{xx}^a} \overline{x^{a2}}) \\ v_{33} &= \hat{\mathbb{V}}(\overline{x^{a2}}) = \frac{1}{N} (\overline{x^{a2}{}^2} - \overline{x^{a2}}^2). \end{aligned}$$

This is the same as what we would get by the sandwich formula. To see this, write the estimated vector ψ for individual i as

$$\hat{\psi}_i = \begin{pmatrix} [(\overline{s_{xx}^a})_i - q\overline{x^{a2}}] \hat{\beta}^c & -(\overline{s_{xy}^a})_i \\ \overline{x^{a2}}_i & -(\overline{x^{a2}})_i \end{pmatrix},$$

from which the inner part of the sandwich follows as

$$\overline{\hat{\psi}\hat{\psi}'} = \begin{pmatrix} \overline{[(s_{xx}^a - qx^{a2})\hat{\beta}^c - s_{xy}^a]^2} & \overline{[(s_{xx}^a - qx^{a2})\hat{\beta}^c - s_{xy}^a][x^{a2} - (x^{a2})]} \\ \overline{[(s_{xx}^a - qx^{a2})\hat{\beta}^c - s_{xy}^a][x^{a2} - (x^{a2})]} & \overline{[x^{a2} - (x^{a2})]^2} \end{pmatrix}.$$

Furthermore,

$$(\hat{\psi}_\theta)_i = \begin{pmatrix} (s_{xx}^a)_i - qx^{a2} & -(1 - \frac{1}{T})\sigma_u^2 \hat{\beta}^c \\ 0 & 1 + \sigma_u^2 \end{pmatrix}$$

and

$$\overline{\hat{\psi}_\theta} = \begin{pmatrix} \overline{s_{xx}^a} - qx^{a2} & -(1 - \frac{1}{T})\sigma_u^2 \hat{\beta}^c \\ 0 & 1 + \sigma_u^2 \end{pmatrix} = \begin{pmatrix} \frac{s_{xy}^a}{\hat{\beta}^c} & -(1 - \frac{1}{T})\sigma_u^2 \hat{\beta}^c \\ 0 & 1 + \sigma_u^2 \end{pmatrix}.$$

It follows that

$$\overline{\hat{\psi}_\theta}^{-1} = \frac{\hat{\beta}^c}{s_{xy}^a(1 + \sigma_u^2)} \begin{pmatrix} 1 + \sigma_u^2 & (1 - \frac{1}{T})\sigma_u^2 \hat{\beta}^c \\ 0 & \frac{s_{xy}^a}{\hat{\beta}^c} \end{pmatrix}$$

With the first unit vector $e' := (1, 0)$, we get

$$e' \overline{\hat{\psi}_\theta}^{-1} = \frac{\hat{\beta}^c}{s_{xy}^a} (1 \quad q\hat{\beta}^c).$$

We thus have all the necessary parts to construct the estimate of the asymptotic variance of $\hat{\beta}^c$:

$$\hat{\mathbb{V}}(\hat{\beta}^c) = \frac{1}{N} e' \overline{\hat{\psi}_\theta}^{-1} \overline{\hat{\psi}\hat{\psi}'} (e' \overline{\hat{\psi}_\theta}^{-1})'.$$

We now only have to further expand the inner part of the sandwich $\overline{\hat{\psi}\hat{\psi}'}$. The upper left corner is given by

$$\begin{aligned} & (\overline{s_{xx}^a} - 2q\overline{s_{xx}^a}x^{a2} + q^2\overline{x^{a2}}^2)\hat{\beta}^{c2} - 2(\overline{s_{xx}^a}s_{xy}^a - qx^{a2}\overline{s_{xy}^a})\hat{\beta}^c + \overline{s_{xy}^a}^2 \\ &= (Nv_{22} + \overline{s_{xx}^a}^2 - 2q\overline{s_{xx}^a}x^{a2} + q^2\overline{x^{a2}}^2)\hat{\beta}^{c2} - 2(Nv_{12} + \overline{s_{xy}^a}s_{xx}^a - qx^{a2}\overline{s_{xy}^a})\hat{\beta}^c + Nv_{11} + \overline{s_{xy}^a}^2 \\ &= Nv_{22}\hat{\beta}^{c2} + (\overline{s_{xx}^a} - qx^{a2})^2\hat{\beta}^{c2} - 2Nv_{12}\hat{\beta}^c - 2(\overline{s_{xx}^a} - qx^{a2})\overline{s_{xy}^a}\hat{\beta}^c + Nv_{11} + \overline{s_{xy}^a}^2 \\ &= Nv_{22}\hat{\beta}^{c2} + s_{xy}^{a2} - 2Nv_{12}\hat{\beta}^c - 2s_{xy}^{a2} + Nv_{11} + s_{xy}^{a2} \\ &= Nv_{22}\hat{\beta}^{c2} - 2Nv_{12}\hat{\beta}^c + Nv_{11} \\ &= N \begin{pmatrix} 1 & -\hat{\beta}^c \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} 1 \\ -\hat{\beta}^c \end{pmatrix}. \end{aligned}$$

Similarly, the right upper and left lower corners are

$$N \begin{pmatrix} v_{13} & v_{23} \end{pmatrix} \begin{pmatrix} 1 \\ -\hat{\beta}^c \end{pmatrix}.$$

Finally the right lower corner is Nv_{33} .

Collecting terms we have

$$\overline{\hat{\psi}\hat{\psi}'} = N \begin{pmatrix} (1 \quad -\hat{\beta}^c) \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} 1 \\ -\hat{\beta}^c \end{pmatrix} & (v_{13} \quad v_{23}) \begin{pmatrix} 1 \\ -\hat{\beta}^c \end{pmatrix} \\ (1 \quad -\hat{\beta}^c) \begin{pmatrix} v_{13} \\ v_{23} \end{pmatrix} & v_{33} \end{pmatrix}$$

and

$$\begin{aligned} \hat{V}(\hat{\beta}^c) &= \frac{\hat{\beta}^{c2}}{S_{xy}^2} (1 \quad q\hat{\beta}^c) \begin{pmatrix} (1 \quad -\hat{\beta}^c) \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} 1 \\ -\hat{\beta}^c \end{pmatrix} & (v_{13} \quad v_{23}) \begin{pmatrix} 1 \\ -\hat{\beta}^c \end{pmatrix} \\ (1 \quad -\hat{\beta}^c) \begin{pmatrix} v_{13} \\ v_{23} \end{pmatrix} & v_{33} \end{pmatrix} \begin{pmatrix} 1 \\ q\hat{\beta}^c \end{pmatrix} \\ &= \frac{\hat{\beta}^{c2}}{S_{xy}^2} (1 \quad -\hat{\beta}^c \quad q\hat{\beta}^c) \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} \begin{pmatrix} 1 \\ -\hat{\beta}^c \\ q\hat{\beta}^c \end{pmatrix}, \end{aligned}$$

which is the variance formula according to the delta method.

Table 2: Simulation results for the iid case - uncorrelated individual effects

ϱ	ρ_{uv}	N	$\hat{\beta}$	$s_{\hat{\beta}}$	$\hat{\beta}^c$	$s_{\hat{\beta}^c}$	$\hat{\sigma}_{\hat{\beta}^c}$	$s_{\hat{\sigma}_{\hat{\beta}^c}}$	$q_{\hat{\beta}^c}^{0.05}$	$q_{\hat{\beta}^c}^{0.50}$	$q_{\hat{\beta}^c}^{0.95}$
T = 3											
-0.50	-0.90	100	0.84283	0.05066	0.99924	0.05891	0.05782	0.00872	0.90154	0.99830	1.09681
		1,000	0.84489	0.01648	0.99997	0.01889	0.01871	0.00100	0.96914	0.99956	1.03066
-0.50	0.00	100	0.91859	0.04105	1.00083	0.04540	0.04439	0.00637	0.92737	1.00093	1.07725
		1,000	0.91875	0.01339	1.00050	0.01475	0.01439	0.00070	0.97608	1.00041	1.02393
-0.50	0.90	100	0.99148	0.02554	0.99966	0.02693	0.02673	0.00354	0.95457	0.99940	1.04433
		1,000	0.99182	0.00800	0.99999	0.00843	0.00858	0.00036	0.98640	0.99997	1.01394
0.00	-0.90	100	0.81154	0.05494	1.00371	0.06589	0.06434	0.00956	0.89923	1.00156	1.11200
		1,000	0.80980	0.01765	1.00030	0.02094	0.02083	0.00112	0.96597	0.99979	1.03525
0.00	0.00	100	0.89991	0.04435	1.00107	0.05037	0.04927	0.00701	0.92314	0.99979	1.08438
		1,000	0.90033	0.01392	1.00044	0.01589	0.01592	0.00076	0.97440	1.00034	1.02637
0.00	0.90	100	0.98919	0.02832	0.99923	0.03047	0.03000	0.00371	0.95045	0.99839	1.04919
		1,000	0.99029	0.00883	1.00030	0.00946	0.00962	0.00039	0.98470	1.00054	1.01586
0.50	-0.90	100	0.69584	0.06767	1.00409	0.09125	0.08734	0.01473	0.86124	1.00040	1.16018
		1,000	0.69625	0.02224	1.00038	0.02893	0.02830	0.00166	0.95370	0.99998	1.04780
0.50	0.00	100	0.84122	0.05552	1.00288	0.06816	0.06568	0.00985	0.89316	1.00238	1.11608
		1,000	0.84001	0.01798	1.00043	0.02185	0.02133	0.00115	0.96501	1.00066	1.03558
0.50	0.90	100	0.98248	0.03604	0.99825	0.04094	0.04056	0.00539	0.93236	0.99821	1.06745
		1,000	0.98331	0.01117	0.99930	0.01280	0.01293	0.00056	0.97738	0.99917	1.02029
T = 10											
-0.50	-0.90	100	0.82120	0.02626	1.00055	0.03112	0.03101	0.00334	0.94925	1.00073	1.05115
		1,000	0.82125	0.00822	1.00010	0.00979	0.00991	0.00037	0.98374	1.00042	1.01594
-0.50	0.00	100	0.90569	0.02138	1.00000	0.02363	0.02341	0.00240	0.96126	0.99982	1.03805
		1,000	0.90610	0.00665	1.00034	0.00745	0.00753	0.00025	0.98817	1.00035	1.01254
-0.50	0.90	100	0.99015	0.01343	0.99960	0.01398	0.01391	0.00131	0.97677	0.99939	1.02260
		1,000	0.99062	0.00421	1.00001	0.00440	0.00443	0.00013	0.99263	1.00004	1.00721
0.00	-0.90	100	0.81098	0.02709	1.00155	0.03199	0.03193	0.00344	0.94922	1.00060	1.05344
		1,000	0.80996	0.00852	0.99984	0.01021	0.01022	0.00035	0.98369	0.99964	1.01726
0.00	0.00	100	0.90057	0.02235	1.00076	0.02511	0.02423	0.00235	0.95949	1.00064	1.04204
		1,000	0.89992	0.00694	0.99994	0.00781	0.00776	0.00025	0.98693	0.99970	1.01265
0.00	0.90	100	0.98946	0.01424	0.99945	0.01490	0.01441	0.00127	0.97498	0.99964	1.02414
		1,000	0.99005	0.00425	1.00010	0.00443	0.00459	0.00013	0.99278	1.00028	1.00704
0.50	-0.90	100	0.77289	0.03006	1.00004	0.03464	0.03513	0.00406	0.94285	0.99969	1.05504
		1,000	0.77404	0.00960	1.00014	0.01135	0.01127	0.00042	0.98176	1.00005	1.01912
0.50	0.00	100	0.88009	0.02384	0.99906	0.02697	0.02662	0.00286	0.95383	0.99827	1.04515
		1,000	0.88097	0.00761	1.00004	0.00866	0.00852	0.00029	0.98526	1.00003	1.01384
0.50	0.90	100	0.98794	0.01498	0.99993	0.01579	0.01593	0.00152	0.97297	0.99996	1.02554
		1,000	0.98815	0.00473	1.00006	0.00499	0.00508	0.00015	0.99181	1.00017	1.00841

Table 3: Simulation results for the iid case - correlated individual effects

ρ	ρ_{uv}	N	$\hat{\beta}$	$s_{\hat{\beta}}$	$\hat{\beta}^c$	$s_{\hat{\beta}^c}$	$\hat{\sigma}_{\hat{\beta}^c}$	$s_{\hat{\sigma}_{\hat{\beta}^c}}$	$q_{\hat{\beta}^c}^{0.05}$	$q_{\hat{\beta}^c}^{0.50}$	$q_{\hat{\beta}^c}^{0.95}$
T = 3											
-0.50	-0.90	100	0.84198	0.05182	1.00353	0.06069	0.05849	0.00894	0.90469	1.00300	1.10751
		1,000	0.84017	0.01667	0.99990	0.01950	0.01895	0.00103	0.96844	0.99998	1.03189
-0.50	0.00	100	0.91949	0.04090	1.00188	0.04542	0.04467	0.00639	0.92735	1.00240	1.07751
		1,000	0.91846	0.01270	1.00007	0.01402	0.01443	0.00073	0.97736	1.00005	1.02415
-0.50	0.90	100	0.99664	0.02597	1.00053	0.02749	0.02642	0.00331	0.95363	1.00012	1.04501
		1,000	0.99621	0.00801	0.99995	0.00847	0.00846	0.00036	0.98647	0.99983	1.01420
0.00	-0.90	100	0.80394	0.05712	1.00761	0.06786	0.06730	0.01061	0.89916	1.00697	1.12481
		1,000	0.79923	0.01864	1.00025	0.02207	0.02176	0.00126	0.96359	1.00036	1.03544
0.00	0.00	100	0.90152	0.04608	1.00247	0.05204	0.05101	0.00759	0.91939	1.00177	1.08949
		1,000	0.89992	0.01491	1.00002	0.01670	0.01643	0.00079	0.97225	0.99988	1.02798
0.00	0.90	100	1.00046	0.02878	0.99962	0.03094	0.03028	0.00400	0.94944	1.00011	1.04933
		1,000	1.00054	0.00904	0.99977	0.00969	0.00972	0.00041	0.98310	0.99993	1.01588
0.50	-0.90	100	0.66539	0.07684	1.00430	0.09926	0.09604	0.01700	0.84435	1.00375	1.17066
		1,000	0.66506	0.02400	1.00162	0.03087	0.03125	0.00207	0.95216	1.00078	1.05388
0.50	0.00	100	0.84124	0.06134	1.00295	0.07481	0.07155	0.01188	0.88265	1.00114	1.12564
		1,000	0.83967	0.01903	0.99996	0.02308	0.02307	0.00131	0.96288	1.00024	1.03774
0.50	0.90	100	1.01583	0.03766	1.00001	0.04313	0.04282	0.00590	0.93001	0.99938	1.07107
		1,000	1.01557	0.01208	0.99985	0.01381	0.01369	0.00063	0.97711	0.99960	1.02227
T = 10											
-0.50	-0.90	100	0.82129	0.02614	1.00235	0.03079	0.03102	0.00343	0.95355	1.00132	1.05371
		1,000	0.81994	0.00867	1.00013	0.01020	0.00996	0.00034	0.98333	0.99981	1.01718
-0.50	0.00	100	0.90609	0.02160	1.00044	0.02402	0.02354	0.00246	0.96159	1.00072	1.03975
		1,000	0.90607	0.00669	1.00027	0.00746	0.00755	0.00025	0.98810	1.00015	1.01259
-0.50	0.90	100	0.99167	0.01378	0.99986	0.01439	0.01385	0.00130	0.97667	0.99972	1.02406
		1,000	0.99166	0.00429	0.99991	0.00453	0.00441	0.00013	0.99240	0.99984	1.00743
0.00	-0.90	100	0.80661	0.02731	0.99959	0.03254	0.03240	0.00370	0.94549	0.99968	1.05442
		1,000	0.80669	0.00868	1.00000	0.01034	0.01035	0.00036	0.98272	0.99983	1.01717
0.00	0.00	100	0.90015	0.02149	1.00043	0.02404	0.02461	0.00262	0.96047	1.00050	1.03960
		1,000	0.89992	0.00690	0.99994	0.00775	0.00784	0.00025	0.98763	0.99980	1.01253
0.00	0.90	100	0.99368	0.01389	1.00044	0.01452	0.01443	0.00131	0.97662	1.00033	1.02445
		1,000	0.99342	0.00440	1.00016	0.00457	0.00460	0.00013	0.99280	1.00009	1.00781
0.50	-0.90	100	0.76291	0.03258	1.00012	0.03670	0.03674	0.00449	0.93984	1.00061	1.06056
		1,000	0.76372	0.01036	1.00007	0.01167	0.01170	0.00046	0.98059	1.00023	1.01986
0.50	0.00	100	0.88159	0.02475	1.00095	0.02822	0.02750	0.00301	0.95519	1.00020	1.04791
		1,000	0.88118	0.00765	1.00037	0.00883	0.00880	0.00031	0.98597	1.00033	1.01485
0.50	0.90	100	0.99787	0.01607	0.99971	0.01683	0.01636	0.00159	0.97226	0.99995	1.02814
		1,000	0.99808	0.00500	0.99991	0.00526	0.00522	0.00016	0.99112	0.99993	1.00860

Table 4: Simulation results for the common factor case - uncorrelated individual effects

ϱ	$\rho_{u^*v^*}$	N	$\hat{\beta}$	$s_{\hat{\beta}}$	$\hat{\beta}^c$	$s_{\hat{\beta}^c}$	$\hat{\sigma}_{\hat{\beta}^c}$	$s_{\hat{\sigma}_{\hat{\beta}^c}}$	$q_{\hat{\beta}^c}^{0.05}$	$q_{\hat{\beta}^c}^{0.50}$	$q_{\hat{\beta}^c}^{0.95}$
T = 3											
-0.50	-0.90	100	0.92168	0.04152	1.00010	0.04460	0.04313	0.00642	0.92761	0.99981	1.07544
		1,000	0.92227	0.01282	1.00024	0.01377	0.01400	0.00069	0.97732	1.00019	1.02295
-0.50	0.00	100	0.95940	0.03429	1.00066	0.03576	0.03480	0.00478	0.94435	1.00066	1.06085
		1,000	0.95913	0.01051	1.00011	0.01099	0.01128	0.00053	0.98239	1.00004	1.01839
-0.50	0.90	100	0.99696	0.02475	1.00112	0.02543	0.02474	0.00317	0.96071	1.00055	1.04261
		1,000	0.99606	0.00766	1.00013	0.00787	0.00796	0.00033	0.98694	1.00001	1.01269
0.00	-0.90	100	0.90301	0.04485	1.00063	0.04874	0.04809	0.00727	0.92040	1.00134	1.08118
		1,000	0.90374	0.01370	1.00017	0.01498	0.01543	0.00075	0.97615	1.00032	1.02483
0.00	0.00	100	0.95125	0.03752	1.00247	0.03966	0.03868	0.00525	0.93777	1.00201	1.06933
		1,000	0.94936	0.01151	1.00017	0.01221	0.01247	0.00054	0.97991	1.00024	1.02007
0.00	0.90	100	0.99619	0.02630	1.00137	0.02722	0.02786	0.00357	0.95706	1.00117	1.04480
		1,000	0.99530	0.00845	1.00038	0.00876	0.00891	0.00037	0.98620	1.00042	1.01513
0.50	-0.90	100	0.84030	0.05733	1.00153	0.06412	0.06369	0.00944	0.89518	1.00084	1.10874
		1,000	0.84089	0.01796	1.00054	0.02013	0.02046	0.00108	0.96704	1.00111	1.03269
0.50	0.00	100	0.91643	0.04770	1.00120	0.05248	0.05084	0.00704	0.91432	1.00064	1.08891
		1,000	0.91629	0.01411	1.00023	0.01556	0.01635	0.00079	0.97557	1.00018	1.02679
0.50	0.90	100	0.99077	0.03442	0.99919	0.03700	0.03694	0.00493	0.93823	0.99973	1.06088
		1,000	0.99157	0.01081	0.99994	0.01158	0.01187	0.00049	0.98069	1.00002	1.01906
T = 10											
-0.50	-0.90	100	0.91033	0.02136	1.00091	0.02312	0.02292	0.00243	0.96371	1.00044	1.03878
		1,000	0.90978	0.00689	1.00021	0.00748	0.00736	0.00025	0.98809	1.00011	1.01232
-0.50	0.00	100	0.95266	0.01726	1.00041	0.01821	0.01842	0.00187	0.97034	1.00104	1.03013
		1,000	0.95234	0.00559	0.99992	0.00586	0.00589	0.00019	0.99030	0.99973	1.00974
-0.50	0.90	100	0.99553	0.01250	1.00030	0.01279	0.01291	0.00126	0.97957	1.00032	1.02092
		1,000	0.99529	0.00390	1.00003	0.00398	0.00413	0.00012	0.99345	1.00000	1.00653
0.00	-0.90	100	0.90314	0.02193	0.99979	0.02380	0.02368	0.00246	0.96016	0.99989	1.03824
		1,000	0.90339	0.00709	0.99977	0.00764	0.00760	0.00025	0.98729	0.99955	1.01254
0.00	0.00	100	0.94956	0.01804	1.00038	0.01908	0.01909	0.00188	0.96886	1.00051	1.03246
		1,000	0.94935	0.00577	1.00005	0.00611	0.00607	0.00019	0.98984	0.99996	1.01049
0.00	0.90	100	0.99476	0.01279	0.99987	0.01320	0.01341	0.00126	0.97766	0.99969	1.02234
		1,000	0.99496	0.00404	1.00002	0.00414	0.00427	0.00012	0.99310	1.00014	1.00687
0.50	-0.90	100	0.88376	0.02337	1.00022	0.02520	0.02647	0.00287	0.95938	1.00069	1.04097
		1,000	0.88413	0.00761	1.00005	0.00819	0.00846	0.00030	0.98662	0.99975	1.01396
0.50	0.00	100	0.93864	0.01981	0.99972	0.02080	0.02096	0.00218	0.96598	0.99918	1.03464
		1,000	0.93892	0.00625	0.99990	0.00666	0.00669	0.00022	0.98905	0.99997	1.01089
0.50	0.90	100	0.99406	0.01409	1.00017	0.01468	0.01486	0.00147	0.97606	0.99993	1.02431
		1,000	0.99393	0.00438	0.99999	0.00452	0.00474	0.00014	0.99271	1.00014	1.00745

Table 5: Simulation results for the common factor case - correlated individual effects

ϱ	$\rho_{u^*v^*}$	N	$\hat{\beta}$	$s_{\hat{\beta}}$	$\hat{\beta}^c$	$s_{\hat{\beta}^c}$	$\hat{\sigma}_{\hat{\beta}^c}$	$s_{\hat{\sigma}_{\hat{\beta}^c}}$	$q_{\hat{\beta}^c}^{0.05}$	$q_{\hat{\beta}^c}^{0.50}$	$q_{\hat{\beta}^c}^{0.95}$
T = 3											
-0.50	-0.90	100	0.92147	0.04153	1.00226	0.04469	0.04362	0.00655	0.93034	1.00198	1.07620
		1,000	0.92008	0.01294	1.00026	0.01389	0.01412	0.00070	0.97797	1.00024	1.02350
-0.50	0.00	100	0.95948	0.03328	1.00078	0.03481	0.03504	0.00474	0.94343	1.00130	1.05946
		1,000	0.95914	0.01066	1.00012	0.01115	0.01132	0.00053	0.98241	0.99988	1.01903
-0.50	0.90	100	0.99822	0.02375	1.00013	0.02447	0.02459	0.00310	0.95904	1.00004	1.04050
		1,000	0.99827	0.00753	1.00014	0.00773	0.00789	0.00033	0.98728	1.00017	1.01262
0.00	-0.90	100	0.89874	0.04683	1.00113	0.05076	0.04919	0.00705	0.91978	1.00086	1.08649
		1,000	0.89814	0.01419	1.00003	0.01551	0.01598	0.00082	0.97471	1.00024	1.02571
0.00	0.00	100	0.94993	0.03779	1.00119	0.03998	0.03965	0.00559	0.93485	1.00088	1.06667
		1,000	0.94944	0.01175	1.00026	0.01246	0.01276	0.00057	0.98002	1.00029	1.02083
0.00	0.90	100	1.00151	0.02688	1.00113	0.02795	0.02777	0.00357	0.95475	1.00097	1.04657
		1,000	1.00074	0.00842	1.00033	0.00874	0.00895	0.00037	0.98610	1.00034	1.01513
0.50	-0.90	100	0.82711	0.06121	1.00472	0.06917	0.06881	0.01187	0.89519	1.00102	1.12371
		1,000	0.82418	0.01950	1.00049	0.02172	0.02218	0.00129	0.96501	1.00143	1.03606
0.50	0.00	100	0.91851	0.04794	1.00388	0.05266	0.05393	0.00791	0.91914	1.00325	1.09186
		1,000	0.91627	0.01494	1.00021	0.01644	0.01726	0.00088	0.97378	1.00000	1.02748
0.50	0.90	100	1.00926	0.03530	1.00098	0.03805	0.03794	0.00490	0.93792	1.00061	1.06190
		1,000	1.00823	0.01124	0.99996	0.01204	0.01222	0.00052	0.97968	0.99981	1.01951
T = 10											
-0.50	-0.90	100	0.90861	0.02181	0.99986	0.02372	0.02301	0.00242	0.96155	1.00017	1.03899
		1,000	0.90922	0.00691	1.00022	0.00746	0.00738	0.00026	0.98817	1.00020	1.01216
-0.50	0.00	100	0.95365	0.01713	1.00137	0.01804	0.01842	0.00191	0.97241	1.00113	1.02991
		1,000	0.95232	0.00564	0.99990	0.00592	0.00589	0.00020	0.99001	0.99996	1.00983
-0.50	0.90	100	0.99577	0.01229	1.00002	0.01260	0.01289	0.00128	0.97931	1.00010	1.02073
		1,000	0.99588	0.00386	1.00004	0.00395	0.00411	0.00012	0.99358	1.00009	1.00681
0.00	-0.90	100	0.90235	0.02172	1.00055	0.02342	0.02410	0.00251	0.96104	1.00052	1.03982
		1,000	0.90173	0.00709	0.99973	0.00759	0.00770	0.00026	0.98714	0.99956	1.01265
0.00	0.00	100	0.94890	0.01841	0.99976	0.01954	0.01917	0.00190	0.96824	0.99959	1.03165
		1,000	0.94931	0.00568	1.00001	0.00601	0.00611	0.00019	0.99018	0.99981	1.00997
0.00	0.90	100	0.99714	0.01293	1.00057	0.01329	0.01337	0.00120	0.97857	1.00047	1.02225
		1,000	0.99662	0.00400	1.00003	0.00412	0.00428	0.00012	0.99315	0.99996	1.00651
0.50	-0.90	100	0.87917	0.02523	1.00056	0.02710	0.02760	0.00325	0.95586	1.00013	1.04470
		1,000	0.87890	0.00811	0.99996	0.00860	0.00881	0.00033	0.98610	1.00010	1.01402
0.50	0.00	100	0.93995	0.01991	1.00102	0.02116	0.02131	0.00226	0.96683	1.00109	1.03580
		1,000	0.93889	0.00639	0.99987	0.00676	0.00684	0.00023	0.98866	0.99995	1.01078
0.50	0.90	100	0.99894	0.01428	0.99981	0.01479	0.01504	0.00147	0.97600	0.99987	1.02427
		1,000	0.99906	0.00447	1.00000	0.00461	0.00481	0.00014	0.99236	1.00012	1.00791