# Cointegration versus Spurious Regression and Heterogeneity in Large Panels

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**Abstract**

This paper provides an estimation and testing framework to identify the source(s) of spuriousness in a large nonstationary panel. This can be determined by two non mutually exclusive causes: pooling units neglecting the presence of heterogeneity and genuine presence of $I(1)$ errors in some of the units. The paper proposes two tests that complement a test for the null of cointegration: one test for the null of homogeneity (and thus presence of spuriousness due to some of the units being genuinely spurious regressions) and one for the null of genuine cointegration in all units of the panel (and thus spuriousness arising only from neglected heterogeneity). The results are derived using a linear combination of two estimators (one consistent, one inconsistent) for the variance of the estimated pooled parameter. The paper also derives two estimators for the degree of heterogeneity and for the fraction of spurious regressions; consistency is achieved as long as $(n, T) \to \infty$, with no need for special restriction on the rate of expansion between $n$ and $T$ as they pass to infinity.

**JEL Codes:** C22, C23.

**Keywords:** Large Panels, Heterogeneity, Spurious Regression.

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# 1 Introduction

Consider the heterogeneous panel regression model

$$y_{it} = \alpha_i + \beta_i x_{it} + u_{it}, \tag{1}$$

where $i = 1, ..., n$, $t = 1, ..., T$ and the variables $y_{it}$ and $x_{it}$ are both $I(1)$ for each $i$. As far as estimation is concerned, sometimes the pooled version of (1) is employed, i.e.

$$y_{it} = \alpha_i + \beta x_{it} + v_{it}, \tag{2}$$

either because the assumption of homogeneity ($\beta_i = \beta$ for all $i$) is not rejected by the data or because the object of interest are not the unit-specific slopes but the long-run average parameter $\beta$ - see also the comments in Temple (1999). However, pooling introduces a further component in the error term, $(\beta_i - \beta) x_{it}$, which is $I(1)$ and thus makes the panel spurious, unless $\beta_i = \beta$; this was noted by Phillips and Moon (1999), who proved that under heterogeneity, (2) is equivalent to a spurious regression and the estimate of $\beta$ is $\sqrt{n}$-consistent as opposed to $\sqrt{n}T$-consistent which would be the case in a cointegrated panel. Thus, imposing homogeneity leads to a spurious panel. On the other hand, in (1), for each $i$ the error term $u_{it}$ can be either $I(0)$, and thus the unit is a cointegration relationship, or $I(1)$, and therefore unit $i$ is genuinely a spurious regression, irrespective of heterogeneity. This situation could e.g. correspond to the case (often found in empirical applications) where $u_{it}$ is truly stationary in accord with some theory, but it is observationally equivalent to an $I(1)$ process due to mis-specification. A comprehensive review of the literature on the possible causes of this is in a recent contribution by Fuertes (2008). Thus, letting $\lambda \in [0, 1]$ be the proportion of units that are spurious regressions, (1) could be a cointegrated panel ($\lambda = 0$), a panel where all units are described by spurious regressions ($\lambda = 1$), or any situation in between. In light of these considerations, a pooled panel model can thus be spurious due to two (not mutually esclusive) reasons: neglected heterogeneity or genuine spuriousness of some units. Therefore, a test for the null of panel cointegration applied to (2) is in fact a test for both homogeneity/poolability and $\lambda = 0$.

The question then arises as to how to disentangle the two possible sources of spuriousness. The purpose of this paper is to provide a step further after rejecting the null of panel cointegration, by providing two tests, one for the null of homogeneity and one for the null of panel cointegration. These two tests make it possible to identify the source of spuriousness in the panel. As a by-product, this paper also proposes two consistent estimators, for the degree of heterogeneity across units and for the fraction of spurious regressions, $\lambda$. The two estimators are consistent and use the model in levels, (2), as opposed to models in differences where the risk of overdifferencing is present. Particularly, the two estimators are based on a linear combination of two estimators, one consistent and one inconsistent, of the variance of the estimate $\hat{\beta}$ in (2). Note that in a mixed panel context (where some units are cointegrated and some are not), estimating the degree of heterogeneity using data in levels is a nontrivial exercise as some of the unit specific estimates $\hat{\beta}_i$s will be inconsistent and therefore cannot be used. Estimation of the fraction of spurious regressions $\lambda$ has been recently addressed by Ng (2008), where a different estimator is proposed to estimate $\lambda \in (0, 1]$. In this paper, a consistent estimator for $\lambda$ is proposed and consistency is shown in all the interval $[0, 1]$, including the boundary $\lambda = 0$; no special assumptions, such as unit long run variances in the $u_{it}$s are required. Based on this estimator, a test for the null of cointegration $H_0 : \lambda = 0$ can be constructed, which has various advantages, since often cointegration is the working hypothesis of interest. Results are derived jointly for $(n, T) \to \infty$, and all the asymptotics is derived allowing for cross dependence of various strength, including strong cross dependence that could arise from a factor structure in the error term.

The paper is organised as follows. Section 2 discusses the model and the main assumptions; consistent estimation of the degree of heterogeneity and of the fraction of spurious regressions is discussed in Section 3, and the results concerning testing are in Section 4. Section 5 concludes.

Notation is fairly standard. Throughout the paper, $\to$ denotes the ordinary limit, $\xrightarrow{d}$ weak convergence and $\xrightarrow[H_0]{d}$ weak convergence under the null hypothesis of a test, and $\xrightarrow{p}$ convergence in probability. Stochastic processes such as $W(r)$ on $[0,1]$ are usually written as $W$, integrals such as $\int_0^1 W(r)\,dr$ as $\int W$; $\bar{W}_1$, $\bar{W}_2$ etc. denote independent demeaned standard Brownian motions. We let $M_1$, $M_2$ etc. such that $M_j < \infty$ be generic positive constants, not depending on $T$ or $n$.

# 2 Model and assumptions

Recall the model (1), where for the sake of simplicity we consider only one regressor, $x_{it}$, and its pooled version

$$\begin{aligned} y_{it} &= \alpha_i + \beta_i x_{it} + u_{it}, \\ y_{it} &= \alpha_i + \beta x_{it} + v_{it}. \end{aligned}$$

We let

$$u_{it} = u_{it}^{(\lambda)} d_\lambda + u_{it}^{(1-\lambda)} (1 - d_\lambda), \tag{3}$$

with $d_\lambda = 1$ for $i = 1, ..., \lfloor n\lambda \rfloor$ and zero otherwise. We assume that $u_{it}^{(\lambda)}$ is non-stationary and $u_{it}^{(1-\lambda)}$ is stationary, i.e.

$$\begin{aligned} u_{it}^{(\lambda)} &= u_{it-1}^{(\lambda)} + \varepsilon_{it}^{u}, \\ \Phi_u(L)\, u_{it}^{(1-\lambda)} &= \varepsilon_{it}^{u}, \end{aligned}$$

where $\Phi_u(L)$ is a lag polynomial with all roots within the unit circle. In (2), it holds that $v_{it} = u_{it} + (\beta_i - \beta)\, x_{it}$. The regressor $x_{it}$ is assumed to be i.i.d. across $i$ have the following DGP

$$x_{it} = x_{it-1} + e_{it}^{x}.$$

Let $\omega_{it} = [e_{it}^{x}, \varepsilon_{it}^{u}]'$ and consider the following assumptions.

**Assumption 1:** [*cross sectional properties*] there exists an invariant $\sigma$-field $C$ such that $E(\omega_{it}|\, C) = 0$ and $\omega_{it}|\, C$ is i.i.d. across $i$.

**Assumption 2:** [*time series properties*] *(i)* $E\,\|\omega_{it}|\, C\|^{8+\delta} < \infty$ for some

$\delta > 0$; *(ii)* and an invariance principle holds for the partial sums of $\omega_{it}$ such that for all $r \in [0,1]$

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor Tr \rfloor} \omega_{it} \overset{a.s.}{=} B_i(r) + O_p\left(\sqrt{\frac{1}{T}}\right),$$

where $B_i(r)$ is a vector Brownian motion with covariance matrix

$$\Omega_i = \begin{bmatrix} \sigma^2_{x,i} & 0 \\ 0 & \sigma^2_{u,i} \end{bmatrix}.$$

**Assumption 3:** [*heterogeneous coefficients*] *(i)* for all $i$, the $\beta_i$s are *i.i.d.* with $E(\beta_i) = \beta$, $Var(\beta_i) = \sigma^2_\beta \in [0, +\infty)$ and $E|\beta_i|^{4+\delta} < \infty$ for some $\delta > 0$; *(ii)* $\{\beta_i\}$ and $\{x_{it}, u_{it}\}$ are two mutually independent groups.

The setup considered here allows for a mixed panel, where $\lfloor n\lambda \rfloor$ units are spurious regressions and the rest of the units are cointegration relationships. Note that allowing for $\lambda \in [0,1]$ means that the boundary cases whereby all units are cointegration/spurious relationships can be accommodated within this framework. Thus, we entertain the cases that *(a)* all units are cointegrated, *(b)* all units are spurious regressions and *(c)* the panel is a mixture of cointegrating and spurious regressions (mixed panel). Hence, the tests discussed below are robust to the boundary cases of cointegration or spurious regression across all units as well as the case of a panel with mixed $I(0)$ and $I(1)$ error terms. Likewise, allowing $\sigma^2_\beta \in [0, +\infty)$ means that the results derived henceforth are valid under both homogeneity and heterogeneity. Note that when $\sigma^2_\beta > 0$ model (2) represents a spurious regression for all units $i$ since the error term is always $I(1)$ because it contains $(\beta_i - \beta)\bar{x}_{it} \sim I(1)$ for all $i$ - see also Phillips and Moon (1999a, p. 1080) - and also possibly because $u_{it} \sim I(1)$ for some $i$.

Assumption 1 considers a general specification for the cross sectional properties of panel $y_{it}$. Cross sectional independence is allowed for, in which case $\omega_{it}| C$ is i.i.d. across $i$ for $C$ being the empty set. On the other extreme, strong cross sectional dependence as could arise form a factor model is considered also. Letting e.g. $\omega_{it} = \omega_{it}^{*} + \lambda_i' f_t$ with $\omega_{it}^{*}$ iid across units and $\lambda_i$ a nonrandom vector of loadings, and considering the $\sigma$-field defined by $\{f_t\}_{t=1}^{T}$, cross sectional independence can be achieved by conditioning on $\{f_t\}_{t=1}^{T}$. A similar argument, to prove the panel asymptotics with common shocks, is also used in Kao, Trapani and Urga (2008). Other forms of cross sectional dependence can be also considered. Note that achieving cross sectional independence by conditioning on some $\sigma$-field is needed to prove the asymptotics hereafter; essentially, the zero mean iid condition assumed in Assumption 1, together with the moment conditions in Assumption 2*(ii)*, make it possible to use a Central Limit Theorem (CLT henceforth) and a Law of Large Numbers (LLN) for Martingale Difference Sequences (MDS). A similar approach was proposed, in a cross sectional framework, by Andrews (2005), and it heavily relies on Hall and Heyde (1980). Last, it is important to note that whilst the presence of a common factor structure is allowed for by Assumption 1, the common factors are required to be $I(0)$. Although some of the asymptotics with $I(1)$ common factors has been discussed in Trapani (2008), in this context it is necessary to rule out the presence of common nonstationary factors, as these would render the error term $u_{it} \sim I(1)$ for all units, thereby leading to $\lambda = 1$ with no need for estimation. A similar argument (with discussion) is also in Ng (2008).

Time dependence is assumed, as long as the Functional Central Limit Theorem holds. Assuming that the remainder is of order $O_p\left(T^{-1/2}\right)$ is quite standard and could e.g. be proved if one assumed a linear process with a Beveridge and Nelson decomposition, by applying the method of proof proposed in Phillips and Solo (1992) - see also Phillips and Moon (1999). The moment condition in Assumption 2*(ii)* is required to prove a Liapunov condition in the asymptotics below.

Assumption 3 is needed only in order for the CLT and the LLN to hold for the $\beta_i$s such that e.g. $n^{-1}\sum_{i=1}^{n}(\beta_i - \beta)^2 \xrightarrow{p} \sigma_\beta^2$ and $\left(n\sigma_\beta^2\right)^{-1/2}\sum_{i=1}^{n}(\beta_i - \beta) \xrightarrow{d} N(0,1)$; less strict assumptions could be considered as long as the CLT and the LLN hold.

Let $\bar{x}_{it} = x_{it} - T^{-1}\sum_t x_{it}$ and $\bar{y}_{it} = y_{it} - T^{-1}\sum_t y_{it}$ and define the LSDV estimator for $\beta$ in (2) as

$$\hat{\beta} = \left[ \sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it}^2 \right]^{-1} \left[ \sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it} \bar{y}_{it} \right],$$

and consider the following (inconsistent and consistent respectively) estimators of the variance of $\hat{\beta}$

$$\begin{aligned} \widehat{Var^*}\left(\hat{\beta}\right) &= \frac{1}{nT} \frac{\sum_{i=1}^{n} \sum_{t=1}^{T} \hat{v}_{it}^2}{\sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it}^2}, \\ \widehat{Var}\left(\hat{\beta}\right) &= \frac{1}{T} \frac{\sum_{i=1}^{n} \left[ \sum_{t=1}^{T} \bar{x}_{it} \hat{v}_{it} \right]^2}{\left[ \sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it}^2 \right]^2} \end{aligned}$$

where $\hat{v}_{it} = \bar{y}_{it} - \hat{\beta}\bar{x}_{it}$. Henceforth, we define $\sigma_x^2 = \lim_{n\to\infty} n^{-1} \sum_i \sigma_{x,i}^2$ and $\sigma_u^2 = \lim_{n\to\infty} \left(\lfloor n\lambda \rfloor\right)^{-1} \sum_{i=1}^{\lfloor n\lambda \rfloor} \sigma_{u,i}^2$. The asymptotics for $\hat{\beta}$, $\widehat{Var}\left(\hat{\beta}\right)$ and $\widehat{Var^*}\left(\hat{\beta}\right)$ is given in the following Theorem.

**Proposition 1** *Let Assumptions 1-3 hold, and assume that the first $\lfloor n\lambda \rfloor$ units are spurious regressions. Then, as $(n, T) \to \infty$ with $\frac{n}{T} \to 0$ it holds that*

$$\sqrt{n}\left(\hat{\beta} - \beta\right) \xrightarrow{d} \sqrt{\frac{2}{5}\frac{\lambda\sigma_u^2}{\sigma_x^2} + \frac{4}{5}\sigma_\beta^2} \times Z, \tag{4}$$

*with $Z \sim N(0, 1)$. As $(n, T) \to \infty$*

$$nT\left[\widehat{Var}\left(\hat{\beta}\right)\right] \xrightarrow{p} \frac{2}{5}\frac{\lambda\sigma_u^2}{\sigma_x^2} + \frac{4}{5}\sigma_\beta^2, \tag{5}$$

$$nT\left[\widehat{Var^*}\left(\hat{\beta}\right)\right] \xrightarrow{p} \frac{\lambda\sigma_u^2}{\sigma_x^2} + \sigma_\beta^2. \tag{6}$$

Proposition 1 states that $\hat{\beta}$ is estimated consistently at a rate $\sqrt{n}$. This result is typical in panel spurious regression as shown by Kao (1999) and Phillips and Moon (1999a). The novel result in (4) is the asymptotic distribution of $\hat{\beta}$ under the broadly general Assumptions 1-3. Note that the limiting distribution of $\hat{\beta}$ is normal instead of mixed normal, contrary to what one might expect in light of Andrews (2005) and Kao, Trapani and Urga (2008a, 2008b). This is due to the assumption that the common factors are stationary, and thus disappear when the terms that contain them are normalised by $T^{-2}$. Note that however, as noted by Kao (1999), $se\left(\hat{\beta}\right)$ is an inconsistent estimator of the variance of $\hat{\beta}$.

Equations (5) and (6) provide the probabilily limits for $\widehat{Var\left(\hat{\beta}\right)}$ and $\widehat{Var^*\left(\hat{\beta}\right)}$. As it can be seen, and as also proved in Trapani (2008) building on an idea in Phillips and Moon (1999), $\widehat{Var\left(\hat{\beta}\right)}$ estimates the asymptotic variance of $\hat{\beta}$ consistently, whilst $\widehat{Var^*\left(\hat{\beta}\right)}$ is an inconsistent estimator. Note that, as it is usually the case for results that involve a LLN as opposed to a CLT, no restrictions on the rate of expansion of $n$ and $T$ is required here.

# 3 Consistent estimation of $\lambda$ and $\sigma_\beta^2$

From (5) and (6), define for brevity the probability limits as

$$\psi_1 = \frac{2}{5}\frac{\lambda\sigma_u^2}{\sigma_x^2} + \frac{4}{5}\sigma_\beta^2, \tag{7}$$

$$\psi_2 = \frac{\lambda\sigma_u^2}{\sigma_x^2} + \sigma_\beta^2. \tag{8}$$

If $\psi_1$ and $\psi_2$ were observable, the estimators of $\lambda$ and $\sigma_\beta^2$ could be expressed as, after solving the linear system defined by (7) and (8)

$$\frac{\lambda\sigma_u^2}{\sigma_x^2} = -\frac{5}{2}\psi_1 + 2\psi_2, \tag{9}$$

$$\sigma_\beta^2 = \frac{5}{2}\psi_1 - \psi_2. \tag{10}$$

Equations (9) and (10) show that (although infeasible) there exists a "direct" estimator for $\sigma_\beta^2$. As far as $\lambda$ is concerned, this cannot be estimated directly, and estimates of $\sigma_x^2$ and of $\sigma_u^2$ are required also. From (5) and (6), consistent estimates for $\psi_1$ and $\psi_2$ are

$$\begin{aligned}\hat{\psi}_1 &= \frac{n\sum_{i=1}^{n}\left[\sum_{t=1}^{T}\bar{x}_{it}\hat{v}_{it}\right]^2}{\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^2\right]^2}, \\ \hat{\psi}_2 &= \frac{\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{v}_{it}^2}{\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^2};\end{aligned}$$

solving the linear system entails

$$\hat{\sigma}_\beta^2 = \frac{5}{2}\hat{\psi}_1 - \hat{\psi}_2. \tag{11}$$

As far as estimation of $\lambda$ is concerned, given an estimate of $\sigma_x^2$, say $\hat{\sigma}_x^2$, after defining

$$\frac{\widehat{\lambda\sigma_u^2}}{\sigma_x^2} = -\frac{5}{2}\hat{\psi}_1 + 2\hat{\psi}_2,$$

it would be possible to estimate directly $\lambda\sigma_u^2$ from (7)-(8) as

$$\widehat{\lambda\sigma_u^2} = \left[-\frac{5}{2}\hat{\psi}_1 + 2\hat{\psi}_2\right]\hat{\sigma}_x^2. \tag{12}$$

A "natural" estimator of $\sigma_x^2$ is given, also in light of (26) in Appendix, by

$$\hat{\sigma}_x^2 = \frac{6}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^2, \tag{13}$$

and we prove that $\hat{\sigma}_x^2 = \sigma_x^2 + O_p\left(T^{-1/2}\right)$, thus being consistent. In order to estimate $\lambda$, we need to filter out $\sigma_u^2$ as well. Note that is is not possible to estimate $\sigma_u^2$ directly, since it is not known a priori which units are spurious regressions and which ones are cointegrated. Thus let $\hat{\beta}_i = \left[\sum_{t=1}^{T}\bar{x}_{it}^2\right]^{-1}\left[\sum_{t=1}^{T}\bar{x}_{it}\bar{y}_{it}\right]$, and define $\hat{u}_{it} = \bar{y}_{it} - \hat{\beta}_i\bar{x}_{it}$. Then a kernel estimator of $\sigma_{u,i}$ can be constructed for each $i$ as $\hat{\sigma}_{u,i} = \hat{\sigma}_{u,i}\left(\hat{u}_{i1}, ..., \hat{u}_{iT}\right)$, and we can define

$$\hat{\psi}_1^{HAC} = \frac{1}{\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^2\right]^2} n \sum_{i=1}^{n}\left[\sum_{t=1}^{T}\bar{x}_{it}\frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}}\right]^2,$$

$$\hat{\psi}_2^{HAC} = \frac{1}{\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}}\right)^2,$$

so that a feasible estimator of $\lambda$ is given by

$$\hat{\lambda} = \left[-\frac{5}{2}\hat{\psi}_1^{HAC} + 2\hat{\psi}_2^{HAC}\right]\hat{\sigma}_x^2. \tag{14}$$

Then it holds that

**Theorem 1** *Let Assumptions 1-3 hold, and assume (for the sake of convenience) that the first $\lfloor n\lambda \rfloor$ units are spurious regressions. Let $d$ be defined as*

$$d = \begin{cases} 1 \text{ if } \lambda = 0 \\ 0 \text{ if } \lambda > 0 \end{cases},$$

*define $d_1 = 1$ if $\lambda = 1$ zero otherwise, and let $d_\sigma = 0$ if $\sigma_\beta^2 = 0$ and $d_\sigma = 1$ otherwise. Assume that $\hat{\sigma}_{u,i} = \sigma_{u,i} + O_p\left(T^{-\varepsilon_1}\right)$ for $i = 1, ..., \lfloor n\lambda \rfloor$ and $\hat{\sigma}_{u,i} = O_p\left(T^{-\varepsilon_2}\right)$ for $i = \lfloor n\lambda \rfloor + 1, ..., n$ for some $\varepsilon_1, \varepsilon_2 \in \left(0, \frac{1}{2}\right)$. Then as $(n, T) \to \infty$*

$$\hat{\sigma}_\beta^2 - \sigma_\beta^2 = O_p\left(\frac{1}{\sqrt{T}}\right) + d_\sigma O_p\left(\frac{1}{\sqrt{n}T^d}\right) + O_p\left(\frac{1}{n}\right) + o_p(1), \tag{15}$$

$$\widehat{\lambda\sigma_u^2} - \lambda\sigma_u^2 = O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{n}}\right) + o_p(1). \tag{16}$$

**Corollary 1** *Under the same assumption as Theorem 1 and using the same notation, it holds that as $(n, T) \to \infty$*

$$\hat{\lambda} - \lambda = (1-d)\, O_p\left(\frac{1}{T^{\varepsilon_1}}\right) + (1-d_1)\, O_p\left(\frac{T^{\varepsilon_2}}{\min\left\{\sqrt{n}, \sqrt{T}\right\}}\right) + o_p(1). \tag{17}$$

Theorem 1 states that $\hat{\sigma}_\beta^2$ and $\widehat{\lambda\sigma_u^2}$ are consistent estimators for $\sigma_\beta^2$ and $\lambda\sigma_u^2$ respectively, as long as $(n,T)\to\infty$. No restrictions on the expansion rate of $n$ and $T$ is required as they pass to infinity: therefore, from a practical viewpoint, the estimators can be applied to panels with all the possible combinations of $n$ and $T$. The Theorem states that $\hat{\sigma}_\beta^2 - \sigma_\beta^2 = O_p\left(1/\min\left\{\sqrt{n},\sqrt{T}\right\}\right)$ for $\lambda\in(0,1]$ and $\sigma_\beta^2>0$, and $\hat{\sigma}_\beta^2 - \sigma_\beta^2 = O_p\left(1/\min\left\{\sqrt{T},n\right\}\right)$ for $\lambda=0$ or $\sigma_\beta^2=0$. Thus, a discontinuity is present in the rate of convergence (and it can also be expected in the limiting distribution) of $\hat{\sigma}_\beta^2$ when one of the two parameters $\left\{\lambda,\sigma_\beta^2\right\}$ is on the boundary. However, contrary to what found in Ng (2008) using a different estimation technique, no discontinuities are found in the rate of convergence of $\widehat{\lambda\sigma_u^2}$ when $\lambda=0$.

Consistent estimation of $\lambda$ is possible according to Corollary 1 ,although at a "slow" rate that depends essentially on $n$ and $T$ and also on the rate of convergence of the HAC estimators $\hat{\sigma}_{u,i}^2$. Note that (17) states that there are discontinuities in the rate of convergence (as found in Ng, 2008) at the boundaries for $\lambda\in[0,1]$. Particularly:

- if $\lambda\in(0,1)$, then

$$\hat{\lambda}-\lambda = O_p\left(\frac{1}{T^{\varepsilon_1}}\right)+O_p\left(\frac{T^{\varepsilon_2}}{\min\left\{\sqrt{n},\sqrt{T}\right\}}\right)+o_p(1);$$

- if $\lambda=0$,

$$\hat{\lambda} = O_p\left(\frac{T^{\varepsilon_2}}{\min\left\{\sqrt{n},\sqrt{T}\right\}}\right)+o_p(1);$$

- finally if $\lambda=1$,

$$\hat{\lambda}-1 = O_p\left(\frac{1}{T^{\varepsilon_1}}\right)+O_p\left(\frac{1}{\sqrt{n}}\right)+o_p(1).$$

The rate of convergence of $\hat{\lambda}$ when $\lambda\in(0,1)$ depends on the speed of convergence of the HAC estimators, $\varepsilon_1$ and $\varepsilon_2$. Assuming ,as it could be expected, that $\varepsilon_1=\varepsilon_2=\varepsilon$, the optimal rate of convergence $\varepsilon$ that maximises the rate of convergence of $\hat{\lambda}$ can be found as a solution of

$$\min_{\varepsilon}\left[\frac{1}{T^{\varepsilon}}+\frac{T^{\varepsilon}}{\min\left\{\sqrt{n},\sqrt{T}\right\}}\right],$$

which after some algebra yields $T^{\varepsilon}=\min\left\{n^{1/4},T^{1/4}\right\}$. This provides an indication as to the optimal choice of the bandwidth when estimating $\sigma_{u,i}^2$.

## 3.1 Limiting distribution for $\hat{\sigma}_{\beta}^2$

As an ancillary result, the following theorem provides the limiting distribution of $\hat{\sigma}_{\beta}^2$ for $\sigma_{\beta}^2\in(0,+\infty)$

**Corollary 2** *Let Assumptions 1-3 hold. As $(n,T)\to\infty$, the limiting distribution of $\sigma_{\beta}^2\in(0,+\infty)$ is given by*

***(i)*** *when $\lambda>0$, under $\frac{n}{T}\to 0$*

$$\sqrt{n}\left(\hat{\sigma}_{\beta}^2-\sigma_{\beta}^2\right)\xrightarrow{d}N\left(0,V_{\sigma}\right), \tag{18}$$

*where*

$$V_{\sigma}=4\left(\lambda\sigma_u^2\right)\sigma_x^2\sigma_{\beta}^2\times\delta_{\sigma}, \tag{19}$$

*with*

$$\delta_{1\sigma}=E\left[\left(\int\bar{W}_1\bar{W}_2\right)^2\left(15\int\bar{W}_1^2-1\right)^2\right]$$

**Lemma 1**

***(ii)*** *when $\lambda=0$, under $\frac{n}{\sqrt{T}}\to 0$*

$$n\left(\hat{\sigma}_{\beta}^2-\sigma_{\beta}^2\right)\xrightarrow{d}\frac{2}{\sqrt{5}}\sigma_{\beta}^2\sigma_x^2\sqrt{\delta_{2\sigma}}\times\chi_{(1)}^2 \tag{20}$$

*where $\chi_{(1)}^2$ is a random variable with a chi-squared distribution with one* *degree of freedom and*

$$\delta_{2\sigma} = E\left[\left(\int \bar{W}^2\right)^2 \left(1 - 30\int \bar{W}^2\right)^2\right].$$

# 4 Tests for cointegration and homogeneity

This section provides the next step forward after applying to (1) a test for the null of panel cointegration. As pointed out in the introduction, (1) can be a spurious panel according to a panel cointegration test due to two reasons (not mutually exclusive), namely that $\lambda = 0$ and/or $\sigma_\beta^2 = 0$. Whilst the former case means that some of the units in the panel are genuinely spurious regression, the latter arises from neglecting heterogeneity after pooling. Formally, a test for the null of cointegration would be based on

$$H_0 : \lambda = 0 \text{ and } \sigma_\beta^2 = 0. \tag{21}$$

When $H_0$ is rejected, then three possible cases can be considered:

1. $\lambda > 0$ and $\sigma_\beta^2 = 0$: the panel is homogeneous and thus pooling is appropriate, but some of the units are (observationally equivalent to) spurious regressions. In this case, a natural further step is the consistent estimation of the fraction of spurious regressions $\lambda$;

2. $\lambda = 0$ and $\sigma_\beta^2 > 0$: the panel is heterogeneous, and thus pooling is not appropriate. All the units in the panel are genuine cointegration relationships, and therefore the null of panel cointegration is rejected simply due to pooling which introduces a nontrivial $I(1)$ component, given by $(\beta_i - \beta)\bar{x}_{it}$, in the error term. A natural step is to avoid pooling, and possibly to estimate the degree of heterogeneity, $\sigma_\beta^2$;

3. $\lambda > 0$ and $\sigma_\beta^2 > 0$: in this case, the panel is heterogeneous and some of the units are spurious regressions. Corollaries 1 and 2 allow to estimate the fraction of spurious regressions in the panel, $\lambda$, and the level of heterogeneity, $\sigma_\beta^2$.

Based on the passages in the proof of Theorem 1, two tests are derived as a follow-up for (21).

### 4.1 Testing for cointegration - $H_0 : \lambda = 0$

The first test which we shall consider is based on

$$\begin{cases} H_0 : \lambda = 0 \text{ and } \sigma^2_\beta \in (0, +\infty) \\ H_A : \lambda > 0 \text{ and } \sigma^2_\beta \in [0, +\infty) \end{cases},$$

i.e. a test whose null hypothesis is that all the units are genuinely cointegrated, and thus spurious regression at a panel level arises from neglecting heterogeneity. Of course under the alternative that some units are spurious regressions, the possibility that the panel is homogeneous ($\sigma^2_\beta = 0$) needs to be entertained also. Note that the null hypothesis in this testing framework is that there is cointegration, which seems natural since cointegration at a micro level is normally assumed in light of some economic theory. Thus, the test is a test for *the null of cointegration.*

The test can be carried out by using $\widehat{\frac{\lambda\sigma^2_u}{\sigma^2_x}}$ instead of $\hat{\lambda}$, which has a slower rate of convergence and for which no distributional results were derived. Since after (16)

$$\widehat{\frac{\lambda\sigma^2_u}{\sigma^2_x}} = \frac{\lambda\sigma^2_u}{\sigma^2_x} + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{n}}\right) + o_p(1),$$

under $H_0 : \lambda = 0$ we have $\widehat{\frac{\lambda\sigma^2_u}{\sigma^2_x}} = O_p\left(T^{-1/2}\right) + O_p\left(n^{-1/2}\right)$. Thus, a natural test statistic for the null of cointegration is (a suitably scaled transformation of) $\min\left\{\sqrt{n}, \sqrt{T}\right\} \times \widehat{\frac{\lambda\sigma^2_u}{\sigma^2_x}}$. Consider also the class of local alternatives

$$H^{(n,T)}_A : \frac{\lambda\sigma^2_u}{\sigma^2_x} = \frac{c}{\min\left\{\sqrt{n}, \sqrt{T}\right\}} \text{ and } \sigma^2_\beta \in [0, +\infty),$$

where $c > 0$. It holds that

**Theorem 2** *Let Assumptions 1-3 hold. Then, as* $(n,T) \to \infty$ *with* $\frac{n}{T} \to 0$

$$\sqrt{n}\left(\widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}}\right) \underset{H_0}{\overset{d}{\to}} \sqrt{\kappa_\beta \times \delta_\lambda} Z, \tag{22}$$

*where* $Z \sim N(0,1)$, $\kappa_\beta = E(\beta_i - \beta)^4$, *and*

$$\delta_\lambda = E\left[\left(\int \bar{W}^2\right)^2 \left(2 - 15\int \bar{W}^2\right)^2\right].$$

*As* $(n,T) \to \infty$ *with* $\frac{T}{n} \to 0$, $\sqrt{T}\left(\widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}}\right) = O_p(1)$. *Letting the limiting distribution of* $\min\left\{\sqrt{n},\sqrt{T}\right\} \times \widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}}$ *as* $(n,T) \to \infty$ *be defined as* $D$, *we have that, under* $H_A^{(n,T)}$ *as* $(n,T) \to \infty$

$$\min\left\{\sqrt{n},\sqrt{T}\right\} \times \widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}} = c + D_\lambda. \tag{23}$$

In light of Theorem 2, a test statistic for the null that $\lambda = 0$ can be constructed as

$$S_{nT}^{(\lambda)} = \sqrt{\frac{n}{\hat{\kappa}_\beta \times \delta_\lambda}}\left(\widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}}\right),$$

where $\hat{\kappa}_\beta$ is a consistent estimate of $\kappa_\beta$; then $S_{nT}^{(\lambda)}$ has a standard normal distribution under $H_0$ as $(n,T) \to \infty$, as long as $\frac{n}{T} \to 0$. Thus, from the practical viewpoint, the test applies to panels where the time series dimension is larger than the cross sectional one. When $\min\left\{\sqrt{n},\sqrt{T}\right\} = \sqrt{T}$, $\sqrt{T}\left(\widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}} - \frac{\lambda\sigma_u^2}{\sigma_x^2}\right) = O_p(1)$; however, this is not a sharp bound and the limiting distribution of this random variable is not standard as it depends on the assumptions on the DGP of the $\bar{x}_{it}$s. According to (23), the test has power versus local alternatives as $(n,T) \to \infty$ for all combinations of $n$ and $T$.

A consistent estimate of $\kappa_\beta$ can be constructed using the micro information as

$$\hat{\kappa}_\beta = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\beta}_i - \widehat{\bar{\beta}}\right)^4,$$

where the $\hat{\beta}_i$s are the estimates from the individual regressions $\Delta y_{it} = \beta_i \Delta x_{it} + \Delta u_{it}$, and $\widehat{\overline{\beta}} = n^{-1} \sum_{i=1}^{n} \hat{\beta}_i$.

## 4.2 Testing for homogeneity: $H_0 : \sigma_\beta^2 = 0$

The second test that is of interest in this context is aimed at verifying whether all the units have the same response to the idiosyncratic covariates $\bar{x}_{it}$, i.e. if $\beta_i = \beta$ for all $i$. Formally, this means

$$\begin{cases} H_0 : \sigma_\beta^2 = 0 \text{ and } \lambda \in (0, 1] \\ H_A : \sigma_\beta^2 > 0 \text{ and } \lambda \in [0, 1] \end{cases}.$$

In this testing framework, $H_0$ states that although some units are cointegrated and other are spurious, the panel is homogeneous, since the slopes $\beta_i$ are the same across all units. Of course, whilst under the null some units must be spurious regressions ($\lambda > 0$), under the alternative we also entertain the possibility that the panel is genuinely cointegrated so that $\lambda = 0$. We also consider the class of local alternatives

$$H_A^{(n,T)} : \sigma_\beta^2 = \frac{c}{\min\left\{n, \sqrt{T}\right\}} \text{ and } \lambda \in [0, 1],$$

with $c > 0$. It holds that

**Theorem 3** *Let Assumptions 1-3 hold. Then, as* $(n, T) \to \infty$ *with* $\frac{n}{\sqrt{T}} \to 0$

$$n\hat{\sigma}_\beta^2 \underset{H_0}{\overset{d}{\to}} \left(6\lambda\sigma_u^2 \sqrt{\frac{\delta_\sigma}{10}}\right) \times \chi_{(1)}^2, \tag{24}$$

*where*

$$\delta_\sigma = E\left[\left(\int \bar{W}_1 \bar{W}_2\right)^2 \left(1 - 10\left(\int \bar{W}_1^2\right)\right)^2\right].$$

*As* $(n, T) \to \infty$ *with* $\frac{\sqrt{T}}{n} \to 0$, $\sqrt{T}\hat{\sigma}_\beta^2 = O_p(1)$. *Letting the limiting distribution of* $\min\left\{n, \sqrt{T}\right\} \times \hat{\sigma}_\beta^2$ *as* $(n, T) \to \infty$ *be defined as* $D_\sigma$, *we have that,* *under* $H_A^{(n,T)}$ *as* $(n,T) \to \infty$

$$\min\left\{n, \sqrt{T}\right\} \times \hat{\sigma}_\beta^2 = c + D_\sigma. \tag{25}$$

Theorem 3 suggests the following test statistic for the null that $\sigma_\beta^2 = 0$

$$S_{nT}^{(\sigma)} = \frac{n}{6\lambda\sigma_u^2}\sqrt{\frac{10}{\delta_\sigma}}\hat{\sigma}_\beta^2;$$

$S_{nT}^{(\sigma)}$ has a chi-squared distribution with one degree of freedom under $H_0$ as $(n,T) \to \infty$, as long as $\frac{n}{\sqrt{T}} \to 0$. The same considerations as for Theorem 2 for the case $\min\left\{n, \sqrt{T}\right\} = \sqrt{T}$ apply here too, and this test too has nontrivial power versus local alternatives. Estimates of $\lambda\sigma_u^2$, $\sigma_x^2$, and $\sigma_\beta^2$, that are needed to make the test feasible, are defined in (12), (13) and (11) respectively.

# 5 Conclusions

This paper provides an estimation and testing framework to identify the source(s) of spuriousness in a large nonstationary panel. This can be determined by two non mutually exclusive causes: pooling units neglecting the presence of heterogeneity and genuine presence of $I(1)$ errors in some of the units. The paper proposes two tests that complement a test for the null of cointegration: one test for the null of homogeneity (and thus presence of spuriousness due to some of the units being genuinely spurious regressions) and one for the null of genuine cointegration in all units of the panel (and thus spuriousness arising only from neglected heterogeneity). The results are derived using a linear combination of two estimators (one consistent, one inconsistent) for the variance of the estimated pooled parameter. The paper also derives two estimators for the degree of heterogeneity and for the fraction of spurious regressions; consistency is achieved as long as $(n,T) \to \infty$, with no need for special restriction on the rate of expansion between $n$ and $T$ as they pass to infinity.

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# 6 Appendix A: useful Lemmas

**Lemma 2** *Let Assumptions 1-3 hold. Then, as $(n,T) \to \infty$, it holds that:*

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^2 \overset{a.s.}{=} \frac{\sigma_x^2}{6} + O_p\left(\frac{1}{\sqrt{T}}\right), \tag{26}$$

$$\frac{1}{nT^4}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2 \xrightarrow{p} \frac{\sigma_x^4}{45}, \tag{27}$$

$$\frac{1}{(n\lambda)\,T^2}\sum_{i=1}^{\lfloor n\lambda\rfloor}\sum_{t=1}^{T}\left[\bar{u}_{it}^{(\lambda)}\right]^2 \overset{a.s.}{=} \frac{\sigma_u^2}{6} + O_p\left(\frac{1}{\sqrt{T}}\right), \tag{28}$$

$$\frac{1}{[n\,(1-\lambda)]\,T^2}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\sum_{t=1}^{T}\left[\bar{u}_{it}^{(1-\lambda)}\right]^2 \overset{a.s.}{=} O_p\left(\frac{1}{T}\right), \tag{29}$$

$$\frac{1}{(n\lambda)\,T^4}\sum_{i=1}^{\lfloor n\lambda\rfloor}\sum_{t=1}^{T}\left[\bar{x}_{it}\bar{u}_{it}^{(\lambda)}\right]^2 \overset{a.s.}{=} \frac{\sigma_x^2\sigma_u^2}{90} + O_p\left(\frac{1}{\sqrt{T}}\right), \tag{30}$$

$$\frac{1}{[n\,(1-\lambda)]\,T^4}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\sum_{t=1}^{T}\left[\bar{x}_{it}\bar{u}_{it}^{(1-\lambda)}\right]^2 \overset{a.s.}{=} O_p\left(\frac{1}{T^2}\right). \tag{31}$$

**Proof.** In order to prove the results in the Lemma, it is not possible to use the central limit theory developed in Phillips and Moon (1999) due to the possible presence of cross sectional dependence among units. The theoretical tools that will be employed are a Law of Large Numbers and a Central Limit Theorem for Martingale Difference Sequences (MDS LLN and MDS CLT henceforth), based on achieving cross sectional independence by conditioning upon the (invariant) $\sigma$-field $C$ - see Assumption 1.

Consider (26), and let $W_{1iT} = T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2$. Conditional on $C$, we have that the $W_{1iT}$s are iid across $i$ in light of Assumption 1. For some constants $M_1, M_2 < \infty$, we have

$$E\left|W_{1iT}\right|C\right| \le M_1 \times E\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right|C\right| \le E\left(\frac{1}{T^2}\sum_{t=1}^{T}\left|\bar{x}_{it}\right|C\right|^2\right).$$

Assumption 2*(ii)* entails that as $T \to \infty$ $T^{-2}\sum_{t=1}^{T} \left|\bar{x}_{it}\right| C|^2 = O_p(1)$ - see Theorem 5.3 in Park and Phillips (1999). Thus, $E\left|W_{1iT}\right| C| < \infty$. As $T \to \infty$, the FCLT entails $W_{1iT}| C = W_{1iT} \overset{a.s.}{=} \sigma_x^2 \int \bar{W}^2 + O_p\left(T^{-1/2}\right)$ and an MDS LLN can be applied whereby

$$\frac{1}{n}\sum_{i=1}^{n} W_{1iT} \overset{a.s.}{=} \sigma_x^2 E\left(\int \bar{W}^2\right) + O\left(\frac{1}{\sqrt{T}}\right);$$

we know from Kao (1999) that $E\left(\int \bar{W}^2\right) = 1/6$. As far as (27) is concerned, define $W_{2iT} = \left(T^{-2}\sum_{t=1}^{T} \bar{x}_{it}^2\right)^2$; then, conditional on $C$, the $W_{2iT}$s are iid across $i$ and for some constants $M_1, M_2 < \infty$

$$E\left|W_{2iT}\right| C| \leq M_1 E\left|\frac{1}{T^4}\sum_{t=1}^{T} \bar{x}_{it}^4\right| C\Bigg| \leq M_2 E\left(\frac{1}{T^4}\sum_{t=1}^{T} \left|\bar{x}_{it}\right| C|^4\right).$$

Similar arguments as before yield $T \to \infty$ $T^{-4}\sum_{t=1}^{T} \left|\bar{x}_{it}\right| C|^4 = O_p(1)$. Thus, $E\left|W_{2iT}\right| C| < \infty$; as $T \to \infty$, the CMT yields $W_{2iT}| C = W_{2iT} \overset{a.s.}{=} \sigma_x^4 \left(\int \bar{W}^2\right)^2 + O_p\left(T^{-1/2}\right)$. Thus, the MDS LLN implies that, as $(n, T) \to \infty$

$$\frac{1}{n}\sum_{i=1}^{n} W_{2iT} \overset{a.s.}{=} \sigma_x^4 E\left[\left(\int \bar{W}^2\right)^2\right] + O\left(\frac{1}{\sqrt{T}}\right).$$

Kao (1999) proves that $E\left[\left(\int \bar{W}^2\right)^2\right] = 1/45$. The proof for (28) is very similar to the proof for (26), and thus it is omitted. As far as (29) is concerned, define $W_{3iT} = T^{-1}\sum_{t=1}^{T}\left[\bar{u}_{it}^{(1-\lambda)}\right]^2$; since conditioning on $C$ the $W_{3iT}$s are independent, and given that, as $T \to \infty$, $W_{3iT}| C = O_p(1)$ for all $i$ in light of the LLN for stationary variables, we have

$$\frac{1}{nT}\sum_{i=1}^{n} W_{3iT} \overset{a.s.}{=} \frac{1}{T} E\left(W_{3iT}| C\right) = O_p\left(\frac{1}{T}\right).$$

We now turn our attention onto (30). Let $W_{4iT} = T^{-4}\left[\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{(\lambda)}\right]^2$; then, conditional on $C$ $W_{4iT}$ is an iid sequence with

$$\begin{aligned}
E\left|W_{4iT}\right|C| &\leq M_1 E\left|\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}^2\left[\bar{u}_{it}^{(\lambda)}\right]^2\right|C\Bigg| \\
&\leq M_1 E\left[\left(\frac{1}{T^4}\sum_{t=1}^{T}\left|\bar{x}_{it}\right|C|^4\right)^{1/2}\left(\frac{1}{T^4}\sum_{t=1}^{T}\left|\bar{u}_{it}^{(\lambda)}\right|C\Big|^4\right)^{1/2}\right],
\end{aligned}$$

and Assumption 2*(ii)* implies that as $T\to\infty$ $T^{-4}\sum_{t=1}^{T}\left|\bar{x}_{it}\right|C|^4 = O_p(1)$ and $T^{-4}\sum_{t=1}^{T}\left|\bar{u}_{it}^{(\lambda)}\right|C\Big|^4 = O_p(1)$. Thus, $E\left|W_{4iT}\right|C| < \infty$; applying the FCLT and the CMT leads to, for $T\to\infty$, $W_{4iT}|C = W_{4iT} \overset{a.s.}{=} \sigma_x^2\sigma_u^2\left(\int\bar{W}_1\bar{W}_2\right)^2 + O_p\left(T^{-1/2}\right)$, and the MDS LLN leads to $n^{-1}\sum_{i=1}^{n}W_{4iT} \overset{a.s.}{=} \sigma_x^2\sigma_u^2 E\left[\left(\int\bar{W}_1\bar{W}_2\right)^2\right]$. Baltagi, Kao and Liu (2008) prove that $E\left[\left(\int\bar{W}_1\bar{W}_2\right)^2\right] = 1/90$. Considering now (31), defining $W_{5iT} = T^{-2}\left[\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{(1-\lambda)}\right]^2$, we have that conditionally on $C$ the $W_{5iT}$s are independent and we could prove that, following similar arguments as for (29) and (30), $E\left|W_{5iT}\right|C| < \infty$. Thus, the MDS LLN implies

$$\frac{1}{nT^2}\sum_{i=1}^{n}W_{5iT} \overset{a.s.}{=} \frac{1}{T^2}E\left(W_{5iT}|C\right) = O_p\left(\frac{1}{T^2}\right).$$

■

# 7 Appendix B: Proofs and derivations

**Proof of Proposition 1.** See Trapani (2008). ■

**Proof of Theorem 1.** Assume, with no loss of generality, that the first $\lfloor n\lambda\rfloor$ units are spurious regressions. In order to derive the results, recall

$$\hat{\beta}-\beta = \left(\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^2\right)^{-1}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it} + \sum_{i=1}^{n}\left(\beta_i-\beta\right)\sum_{t=1}^{T}\bar{x}_{it}^2\right] = O_p\left(\frac{1}{\sqrt{n}}\right),$$

and consider the following expansions:

$$
\begin{aligned}
& \sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\hat{v}_{it}\right)^{2} \\
= \;& \sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^{2}+\sum_{i=1}^{n}\left[\left(\beta_{i}-\beta\right)^{2}\left(\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)^{2}\right]+\left(\hat{\beta}-\beta\right)^{2}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)^{2} \\
& +2\sum_{i=1}^{n}\left[\left(\beta_{i}-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right]-2\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\left[\left(\beta_{i}-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)^{2}\right] \\
& -2\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\left[\left(\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right], \qquad (32)
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{i=1}^{n}\sum_{t=1}^{T}\hat{v}_{it}^{2} \\
= \;& \sum_{i=1}^{n}\sum_{t=1}^{T}\bar{u}_{it}^{2}+\sum_{i=1}^{n}\left[\left(\beta_{i}-\beta\right)^{2}\left(\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)\right]+\left(\hat{\beta}-\beta\right)^{2}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^{2} \\
& +2\sum_{i=1}^{n}\left[\left(\beta_{i}-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right]-2\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it} \\
& -2\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\left[\left(\beta_{i}-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)\right]. \qquad (33)
\end{aligned}
$$

Consider first (15), and consider the quantity $\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^{2}$. It is well known, e.g. from Phillips and Moon (1999), that $\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^{2}=O_{p}\left(nT^{2}\right)$; Kao (1999) proves that, as $(n,T)\rightarrow\infty$, $\left(nT^{2}\right)^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^{2}=\sigma_{x}^{2}/6+o_{p}(1)$. Replacing $\left(nT^{2}\right)^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^{2}$ with its limit, using (32) and (33) after some algebra we have

$$
\begin{aligned}
\hat{\sigma}_\beta^2 &= \frac{5}{2}\hat{\psi}_1 - \hat{\psi}_2 \\
&= \frac{6}{\sigma_x^2}\left[\frac{15}{\sigma_x^2 nT^4}\sum_{i=1}^n\left(\sum_{t=1}^T \bar{x}_{it}\hat{v}_{it}\right)^2 - \frac{1}{nT^2}\sum_{i=1}^n\sum_{t=1}^T \hat{v}_{it}^2\right] \\
&= \frac{6}{\sigma_x^2}\left\{\frac{15}{\sigma_x^2 nT^4}\sum_{i=1}^n\left(\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)^2 - \frac{1}{nT^2}\sum_{i=1}^n\sum_{t=1}^T \bar{u}_{it}^2\right. \\
&\quad + \frac{1}{nT^2}\sum_{i=1}^n(\beta_i-\beta)^2\left[\frac{15}{\sigma_x^2 T^2}\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2 - \left(\sum_{t=1}^T \bar{x}_{it}^2\right)\right] \\
&\quad + \frac{1}{nT^2}\left(\hat{\beta}-\beta\right)^2\sum_{i=1}^n\left[\frac{15}{\sigma_x^2 T^2}\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2 - \left(\sum_{t=1}^T \bar{x}_{it}^2\right)\right] \\
&\quad + \frac{30}{\sigma_x^2 nT^4}\sum_{i=1}^n\left[(\beta_i-\beta)\left(\sum_{t=1}^T \bar{x}_{it}^2\right)\left(\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)\right] \\
&\quad - \frac{30}{\sigma_x^2 nT^4}\left(\hat{\beta}-\beta\right)\sum_{i=1}^n\left[(\beta_i-\beta)\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2\right] \\
&\quad - \frac{30}{\sigma_x^2 nT^4}\left(\hat{\beta}-\beta\right)\sum_{i=1}^n\left[\left(\sum_{t=1}^T \bar{x}_{it}^2\right)\left(\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)\right] - \frac{2}{nT^2}\sum_{i=1}^n\left[(\beta_i-\beta)\left(\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)\right] \\
&\quad \left. + \frac{2}{nT^2}\left(\hat{\beta}-\beta\right)\left(\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right) + \frac{2}{nT^2}\left(\hat{\beta}-\beta\right)\sum_{i=1}^n\left[(\beta_i-\beta)\left(\sum_{t=1}^T \bar{x}_{it}^2\right)\right]\right\} \\
&= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10}. \qquad (34)
\end{aligned}
$$

Consider $A_1$:

$$
\begin{aligned}
A_1 &= \frac{15}{\sigma_x^2 n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{1}{T^2}\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)^2 + \frac{15}{\sigma_x^2 nT^2}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\frac{1}{T}\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)^2 \\
&= A_{11} + A_{12}.
\end{aligned}
$$

As $(n,T)\to\infty$, (26) yields

$$A_{11}=15\times\frac{\lambda\sigma_u^2}{90}+O_p\left(\frac{1}{\sqrt{T}}\right),$$

where the term $O_p\left(T^{-1/2}\right)$ is a sampling error - see also Phillips and Moon (1999); also, $A_{12}=O_p\left(T^{-2}\right)$. Similarly, as far as $A_2$ is concerned, we have

$$\begin{aligned}-A_2 &= \frac{1}{nT^2}\sum_{i=1}^{\lfloor n\lambda\rfloor}\sum_{t=1}^{T}\bar{u}_{it}^2+\frac{1}{nT^2}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\sum_{t=1}^{T}\bar{u}_{it}^2\\ &= A_{21}+A_{22},\end{aligned}$$

and as $(n,T)\to\infty$

$$A_{21}=\frac{\lambda\sigma_u^2}{6}+O_p\left(\frac{1}{\sqrt{T}}\right),$$

whilst $A_{22}=O_p\left(T^{-1}\right)$. Thus, $A_1+A_2=O_p\left(T^{-1/2}\right)$. As far as $A_3$ is concerned, define

$$Z_{1iT}=\frac{15}{\sigma_x^2T^4}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2-\frac{1}{T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right);$$

Assumption 2 ensures that $Z_{1iT}$ is independent of $(\beta_i-\beta)$; as $T\to\infty$, we have

$$E\left(Z_{1iT}\right)=\frac{1}{3}\sigma_x^2-\frac{1}{6}\sigma_x^2+O_p\left(\frac{1}{\sqrt{T}}\right)=\frac{1}{6}\sigma_x^2+O_p\left(\frac{1}{\sqrt{T}}\right),$$

where the term $O_p\left(T^{-1/2}\right)$ arises from a sampling error. Thus, the sequence $(\beta_i-\beta)^2 Z_{1iT}$ is an iid sequence with $E\left[(\beta_i-\beta)^2 Z_{1iT}\right]=E\left[(\beta_i-\beta)^2\right]E\left[Z_{1iT}\right]=\sigma_\beta^2\sigma_x^2/6+O_p\left(T^{-1/2}\right)$; the MDS LLN entails

$$A_3\overset{a.s.}{=}\frac{\sigma_\beta^2\sigma_x^2}{6}+O_p\left(\frac{1}{\sqrt{T}}\right).$$

Considering $A_4$, we have

$$A_4=\frac{1}{n}\left(\hat{\beta}-\beta\right)^2\sum_{i=1}^{n}Z_{1iT},$$

where $Z_{1iT}$ is a sequence of nonzero mean, iid variables and the MDS LLN yields $\sum_{i=1}^{n} Z_{1iT} = O_p(n)$. Since $\hat{\beta} - \beta = O_p\left(n^{-1/2}\right)$, we have $A_4 = O_p\left(n^{-1}\right)$. As far as $A_5$ is concerned, we have

$$
\begin{aligned}
\frac{30}{\sigma_x^2 n} \sum_{i=1}^{n} \left(\beta_i - \beta\right) Z_{2iT} &= \frac{30}{\sigma_x^2 n} \sum_{i=1}^{\lfloor n\lambda \rfloor} \left(\beta_i - \beta\right) Z_{2iT} + \frac{30}{\sigma_x^2 n} \sum_{i=\lfloor n\lambda \rfloor + 1}^{n} \left(\beta_i - \beta\right) Z_{2iT} \\
&= A_{51} + A_{52},
\end{aligned}
$$

where $Z_{2iT} = \left(T^{-2} \sum_{t=1}^{T} \bar{x}_{it}^2\right)\left(T^{-2} \sum_{t=1}^{T} \bar{x}_{it} \bar{u}_{it}\right)$. Note that the sequence $\left(\beta_i - \beta\right) Z_{2iT}$ is iid zero mean, and therefore a CLT ensures that $A_{51} = O_p\left(n^{-1/2}\right)$; as far as $A_{52}$ is concerned, $Z_{2iT} = O_p\left(T^{-1}\right)$ and therefore $A_{52} = O_p\left(\frac{1}{\sqrt{n}T}\right)$. Thus, $A_5 = d_\sigma O_p\left(\frac{1}{\sqrt{n}T^d}\right)$. Consider now $A_6$; defining $Z_{3iT} = T^{-4}\left(\sum_{t=1}^{T} \bar{x}_{it}^2\right)^2$, the sequence $\left(\beta_i - \beta\right) Z_{3iT}$ is iid zero mean and thus

$$
\begin{aligned}
-A_6 &= \frac{30}{\sigma_x^2 n}\left(\hat{\beta} - \beta\right) \sum_{i=1}^{n} \left(\beta_i - \beta\right) Z_{3iT} \\
&= O_p\left(\frac{1}{\sqrt{n}}\right) d_\sigma O_p\left(\frac{1}{\sqrt{n}}\right) = d_\sigma O_p\left(\frac{1}{n}\right).
\end{aligned}
$$

As far as $A_7$ is concerned, define $Z_{4iT} = \left(T^{-2} \sum_{t=1}^{T} \bar{x}_{it}^2\right)\left(T^{-2} \sum_{t=1}^{T} \bar{x}_{it} \bar{u}_{it}\right)$, so that

$$
\begin{aligned}
-A_7 &= \frac{30}{\sigma_x^2 n}\left(\hat{\beta} - \beta\right) \sum_{i=1}^{\lfloor n\lambda \rfloor} Z_{4iT} + \left(\hat{\beta} - \beta\right) \frac{30}{\sigma_x^2 n} \sum_{i=\lfloor n\lambda \rfloor + 1}^{n} Z_{4iT} \\
&= A_{71} + A_{72};
\end{aligned}
$$

as $T \to \infty$, $E\left(Z_{4iT}\right) = O\left(T^{-1/2}\right)$ and thus

$$
\begin{aligned}
\sum_{i=1}^{\lfloor n\lambda \rfloor} Z_{4iT} &= \sum_{i=1}^{\lfloor n\lambda \rfloor} \left[Z_{4iT} - E\left(Z_{4iT}\right)\right] + \sum_{i=1}^{\lfloor n\lambda \rfloor} E\left(Z_{4iT}\right) = O_p\left(\sqrt{n}\right) + O\left(\frac{n}{\sqrt{T}}\right), \\
\sum_{i=\lfloor n\lambda \rfloor + 1}^{n} Z_{4iT} &= \sum_{i=\lfloor n\lambda \rfloor + 1}^{n} \left[Z_{4iT} - E\left(Z_{4iT}\right)\right] + \sum_{i=\lfloor n\lambda \rfloor + 1}^{n} E\left(Z_{4iT}\right) = O_p\left(\frac{\sqrt{n}}{T}\right) + O\left(\frac{n}{T^{3/2}}\right).
\end{aligned}
$$

Thus, $A_7 = O_p\left(\frac{1}{nT^d}\right) + O_p\left(\frac{1}{\sqrt{nTT^d}}\right)$. Considering $A_8$, define $Z_{5iT} = T^{-2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}$; we have

$$
\begin{aligned}
-A_8 &= \frac{2}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\beta_i - \beta\right) Z_{5iT} + \frac{2}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\beta_i - \beta\right) Z_{5iT} \\
&= A_{81} + A_{82};
\end{aligned}
$$

since $\left(\beta_i - \beta\right) Z_{5iT}$ is iid zero mean, we have $A_{81} = O_p\left(n^{-1/2}\right)$ and $A_{82} = O_p\left(\frac{1}{\sqrt{nT}}\right)$, so that $A_8 = d_\sigma O_p\left(\frac{1}{\sqrt{nT^d}}\right)$. As far as $A_9$ is concerned, note that

$$
\begin{aligned}
A_9 &= \frac{2}{n}\left(\hat{\beta} - \beta\right)\sum_{i=1}^{\lfloor n\lambda\rfloor} Z_{5iT} + \frac{2}{n}\left(\hat{\beta} - \beta\right)\sum_{i=\lfloor n\lambda\rfloor+1}^{n} Z_{5iT} \\
&= A_{91} + A_{92};
\end{aligned}
$$

as $T \to \infty$, we have $E\left(Z_{5iT}\right) = O\left(T^{-1/2}\right)$, and therefore

$$
\begin{aligned}
A_{91} &= \frac{2}{n}\left(\hat{\beta} - \beta\right)\sum_{i=1}^{\lfloor n\lambda\rfloor}\left[Z_{5iT} - E\left(Z_{5iT}\right)\right] + \frac{2}{n}\left(\hat{\beta} - \beta\right)\sum_{i=1}^{\lfloor n\lambda\rfloor} E\left(Z_{5iT}\right) \\
&= O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{nT}}\right);
\end{aligned}
$$

similar arguments lead to $A_{92} = O_p\left(\frac{1}{nT}\right) + O_p\left(\frac{1}{\sqrt{nTT}}\right)$. Hence, $A_9 = O_p\left(\frac{1}{nT^d}\right) + O_p\left(\frac{1}{\sqrt{nTT^d}}\right)$. Last, as far as $A_{10}$ is concerned, letting $Z_{6iT} = \left(\beta_i - \beta\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)$, we have $E\left(Z_{6iT}\right) = 0$, and $E\left|Z_{6iT}\right|^{2+\delta} < \infty$; thus, the MDS CLT yields $\sum_{i=1}^{n} Z_{6iT} = O_p\left(\sqrt{n}\right)$ and therefore $A_{10} = d_\sigma O_p\left(\frac{1}{n}\right)$. Putting all together, we have

$$
\hat{\sigma}_\beta^2 = \frac{6}{\sigma_x^2} \times \frac{\sigma_\beta^2\sigma_x^2}{6} + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{n}\right) + d_\sigma O_p\left(\frac{1}{\sqrt{nT^d}}\right) + o_p(1). \quad (35)
$$

The proof for (16) follows similar passages. We have

$$
\widehat{\lambda\sigma_u^2} = \left[-\frac{5}{2}\hat{\psi}_1 + 2\hat{\psi}_2\right]\hat{\sigma}_x^2,
$$

and replacing $(nT^2)^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^2$ and $\hat{\sigma}_x^2$ with their limit, using (32) and (33) after some algebra we have

$$
\begin{aligned}
\widehat{\lambda\sigma_u^2} &= 6\left[-\frac{15}{\sigma_x^2 nT^4}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\hat{v}_{it}\right)^2+\frac{2}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{v}_{it}^2\right] \\
&= 6\left\{-\frac{15}{\sigma_x^2 nT^4}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2+\frac{2}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{u}_{it}^2\right. \\
&\quad +\frac{1}{nT^2}\sum_{i=1}^{n}(\beta_i-\beta)^2\left[2\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)-\frac{15}{\sigma_x^2T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\right] \\
&\quad +\frac{1}{nT^2}\left(\hat{\beta}-\beta\right)^2\sum_{i=1}^{n}\left[2\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)-\frac{15}{\sigma_x^2T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\right] \\
&\quad -\frac{30}{\sigma_x^2 nT^4}\sum_{i=1}^{n}\left[(\beta_i-\beta)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right] \\
&\quad +\frac{30}{\sigma_x^2 nT^4}\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\left[(\beta_i-\beta)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\right] \\
&\quad +\frac{30}{\sigma_x^2 nT^4}\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\left[\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right]+\frac{4}{nT^2}\sum_{i=1}^{n}\left[(\beta_i-\beta)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right] \\
&\quad \left.-\frac{4}{nT^2}\left(\hat{\beta}-\beta\right)\left(\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)-\frac{4}{nT^2}\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\left[(\beta_i-\beta)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\right]\right\} \\
&= B_1+B_2+B_3+B_4+B_5+B_6+B_7+B_8+B_9+B_{10}. \qquad (36)
\end{aligned}
$$

We have, in light of similar arguments as before, that as $(n,T)\to\infty$

$$
\begin{aligned}
B_1 &= -15\times\frac{\lambda\sigma_u^2}{90}+O_p\left(\frac{1}{\sqrt{T}}\right), \\
B_2 &= 2\times\frac{\lambda\sigma_u^2}{6}+O_p\left(\frac{1}{\sqrt{T}}\right),
\end{aligned}
$$

and therefore

$$
B_1+B_2=\frac{\lambda\sigma_u^2}{6}+O_p\left(\frac{1}{\sqrt{T}}\right).
$$

Consider $B_3$, and define

$$Z_{6iT} = \frac{2}{T^2}\left(\sum_{t=1}^{T} \bar{x}_{it}^2\right) - \frac{15}{\sigma_x^2 T^4}\left(\sum_{t=1}^{T} \bar{x}_{it}^2\right)^2;$$

as $T \to \infty$, we have

$$\begin{aligned} E\left(Z_{6iT}\right) &= 2 \times \frac{\sigma_x^2}{6} - \frac{15}{\sigma_x^2} \times \frac{\sigma_x^4}{45} + O_p\left(\frac{1}{\sqrt{T}}\right) \\ &= O_p\left(\frac{1}{\sqrt{T}}\right). \end{aligned}$$

Hence

$$\begin{aligned} B_3 &= \frac{1}{n}\sum_{i=1}^{n} \left(\beta_i - \beta\right)^2 \left[Z_{6iT} - E\left(Z_{6iT}\right)\right] + \frac{1}{n}\sum_{i=1}^{n} \left(\beta_i - \beta\right)^2 E\left(Z_{6iT}\right) \\ &= B_{31} + B_{32}, \end{aligned}$$

and since $\left(\beta_i - \beta\right)^2 \left[Z_{6iT} - E\left(Z_{6iT}\right)\right]$ is an iid zero mean sequence, the CLT entails $B_{31} = O_p\left(n^{-1/2}\right)$. Also, $B_{32} = O_p\left(T^{-1/2}\right)$. Thus,

$$B_3 = O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right).$$

As far as $B_4$ is concerned, we have

$$\begin{aligned} B_4 &= \left(\hat{\beta} - \beta\right)^2 \frac{1}{n}\sum_{i=1}^{n} Z_{6iT} \\ &= \left(\hat{\beta} - \beta\right)^2 \frac{1}{n}\sum_{i=1}^{n} \left[Z_{6iT} - E\left(Z_{6iT}\right)\right] + \left(\hat{\beta} - \beta\right)^2 \frac{1}{n}\sum_{i=1}^{n} E\left(Z_{6iT}\right) \\ &= B_{41} + B_{42}; \end{aligned}$$

similar arguments as above, together with $\hat{\beta} - \beta = O_p\left(n^{-1/2}\right)$, lead to

$$B_4 = O_p\left(\frac{1}{n\sqrt{n}}\right) + O_p\left(\frac{1}{n\sqrt{T}}\right).$$

As far as the other terms are concerned, same arguments as in the proof for (15) yield: $B_5 = O_p\left(\frac{1}{\sqrt{n}T^d}\right)$; $B_6 = O_p\left(n^{-1}\right)$; $B_7 = O_p\left(\frac{1}{nT^d}\right) + O_p\left(\frac{1}{\sqrt{n}TT^d}\right)$; $B_8 = O_p\left(\frac{1}{\sqrt{n}T^d}\right)$; $B_9 = B_{10} = O_p\left(\frac{1}{n}\right)$. Putting all together, we have

$$\widehat{\lambda\sigma_u^2} = 6 \times \frac{\lambda\sigma_u^2}{6} + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{n}}\right) + o_p(1). \tag{37}$$

■

**Proof of Corollary 1.** The proof follows a slightly different approach to Theorem 1, since HAC residuals are involved and thus the estimation errors $\hat{\sigma}_{u,i}^2 - \sigma_{u,i}^2$ enter $\hat{\lambda} - \lambda$. Recall the definition of $\hat{\lambda}$

$$\begin{aligned} \hat{\lambda} &= -\frac{5}{2}\frac{n}{\left[\sum_{i=1}^n \sum_{t=1}^T \bar{x}_{it}^2\right]^2} \sum_{i=1}^n \left(\sum_{t=1}^T \bar{x}_{it}\frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}}\right)^2 \times \frac{6}{nT^2}\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}^2 \\ &\quad +2\frac{1}{\sum_{i=1}^n \sum_{t=1}^T \bar{x}_{it}^2}\sum_{i=1}^n\sum_{t=1}^T \left(\frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}}\right)^2 \times \frac{6}{nT^2}\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}^2 \\ &= -\frac{15}{T^2\sum_{i=1}^n \sum_{t=1}^T \bar{x}_{it}^2} \times \sum_{i=1}^n \left(\sum_{t=1}^T \bar{x}_{it}\frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}}\right)^2 + \frac{12}{nT^2}\sum_{i=1}^n\sum_{t=1}^T \left(\frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}}\right)^2 \end{aligned} \tag{38}$$

Replacing $\left(nT^2\right)^{-1}\sum_{i=1}^n \sum_{t=1}^T \bar{x}_{it}^2$ with its limit (and omitting higher order terms for the sake of the notation), defining

$$\hat{v}_{it}^+ = \frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}},$$

and $\bar{u}_{it}^+ = \bar{u}_{it}/\hat{\sigma}_{u,i}$, and recalling that $\hat{v}_{it} = \bar{u}_{it} + (\beta_i - \beta)\bar{x}_{it} - \left(\hat{\beta} - \beta\right)\bar{x}_{it}$,

(38) can be further expressed as

$$
\begin{aligned}
\hat{\lambda} &= -\frac{90}{nT^4\sigma_x^2} \times \sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\hat{v}_{it}^{+}\right)^2 + \frac{12}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{v}_{it}^{+2} \\
&= -\frac{90}{nT^4\sigma_x^2}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)^2 - \frac{90}{nT^4\sigma_x^2}\sum_{i=1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\right] \\
&\quad -\frac{90\left(\hat{\beta}-\beta\right)^2}{nT^4\sigma_x^2}\sum_{i=1}^{n}\left(\frac{1}{\hat{\sigma}_{u,i}}\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2 - \frac{180\left(\hat{\beta}-\beta\right)}{nT^4\sigma_x^2}\sum_{i=1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\right] \\
&\quad +\frac{180}{nT^4\sigma_x^2}\sum_{i=1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\right] \\
&\quad -\frac{180\left(\hat{\beta}-\beta\right)}{nT^4\sigma_x^2}\sum_{i=1}^{n}\left[\frac{1}{\hat{\sigma}_{u,i}}\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\right] + \frac{12}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{u}_{it}^{+2} \\
&\quad +\frac{12}{nT^2}\sum_{i=1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2\sum_{t=1}^{T}\bar{x}_{it}^2\right] + \frac{12\left(\hat{\beta}-\beta\right)^2}{nT^2}\sum_{i=1}^{n}\left[\frac{1}{\hat{\sigma}_{u,i}^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right] \\
&\quad +\frac{24}{nT^2}\sum_{i=1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)\right] - \frac{24\left(\hat{\beta}-\beta\right)}{nT^2}\sum_{i=1}^{n}\left[\frac{1}{\hat{\sigma}_{u,i}}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right] \\
&\quad -\frac{24\left(\hat{\beta}-\beta\right)}{nT^2}\sum_{i=1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}^2}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\right] \\
&= C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8 + C_9 + C_{10} + C_{11} + C_{12}.
\end{aligned}
$$

Consider $C_1$. We have

$$
\begin{aligned}
-C_1 &= \frac{90}{n\sigma_x^2}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)^2 + \frac{90}{n\sigma_x^2}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)^2 \\
&= C_{1,1} + C_{1,2}.
\end{aligned}
$$

As far as $C_{1,1}$ is concerned, it holds that

$$\begin{aligned}
C_{1,1} &= \frac{90}{n\sigma_x^2}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}\frac{\bar{u}_{it}}{\sigma_{u,i}}\right)^2 - \frac{90}{n\sigma_x^2}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{\hat{\sigma}_{u,i}^2-\sigma_{u,i}^2}{\sigma_{u,i}^2}\right)\left(\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}\frac{\bar{u}_{it}}{\sigma_{u,i}}\right)^2 \\
&= C_{1,1,1}+C_{1,1,2};
\end{aligned}$$

we know that $C_{1,1,1}=\lambda+O_p\left(T^{-1/2}\right)$; as far as $C_{1,1,2}$ is concerned, we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{\hat{\sigma}_{u,i}^2-\sigma_{u,i}^2}{\sigma_{u,i}^2}\right)\left(\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}\frac{\bar{u}_{it}}{\sigma_{u,i}}\right)^2 &\leq \max_i\left|\frac{\hat{\sigma}_{u,i}^2-\sigma_{u,i}^2}{\sigma_{u,i}^2}\right|\times\frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}\frac{\bar{u}_{it}}{\sigma_{u,i}}\right)^2 \\
&= O_p\left(\frac{1}{T^{\varepsilon_1}}\right).
\end{aligned}$$

As far as $C_{1,2}$ is concerned, note that

$$\begin{aligned}
C_{1,2} &= \frac{90}{n\sigma_x^2}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}\frac{\bar{u}_{it}}{\hat{\sigma}_{u,i}}\right)^2 \\
&\leq \frac{90}{n\sigma_x^2}\max_i\left|\frac{1}{\hat{\sigma}_{u,i}^2}\right|\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2 = O_p\left(T^{\varepsilon_2}\right)O_p\left(\frac{1}{T^2}\right) = O_p\left(T^{\varepsilon_2-2}\right).
\end{aligned}$$

Thus, $C_1=-\lambda+O_p\left(T^{-1/2}\right)+O_p\left(T^{-\varepsilon_1}\right)+O_p\left(T^{\varepsilon_2-2}\right)$; clearly, the term of magnitude $O_p\left(T^{-\varepsilon_1}\right)$ is present only if $\lambda>0$. Consider now $C_2+C_8$

$$C_2+C_8=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2\left[-\frac{90}{\sigma_x^2T^4}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2+\frac{12}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right],$$

and letting

$$X_{1iT}=-\frac{90}{\sigma_x^2T^4}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2+\frac{12}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2,$$

we can write

$$
\begin{aligned}
C_2+C_8 &= \frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2 X_{1iT}\right]+\frac{1}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2 X_{1iT}\right] \\
&= C_{28,1}+C_{28,2}.
\end{aligned}
$$

Since $E\left(X_{1iT}\right)=O_p\left(T^{-1/2}\right)$, defining $\bar{X}_{1iT}=X_{1iT}-E\left(X_{1iT}\right)$ we have

$$
\begin{aligned}
C_{28,1} &= \frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2 \bar{X}_{1iT}+\frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2 E\left(X_{1iT}\right)\right] \\
&= C_{28,1,1}+C_{28,1,2};
\end{aligned}
$$

the MDS CLT entails that $C_{28,1,1}=O_p\left(n^{-1/2}\right)$, and

$$
C_{28,1,2}\leq \max_i\left|E\left(X_{1iT}\right)\right|\frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2=O_p\left(\frac{1}{\sqrt{T}}\right).
$$

As far as $C_{28,2}$ is concerned, we have

$$
\begin{aligned}
C_{28,2} &\leq \max_i\left(\frac{1}{\hat{\sigma}^2_{u,i}}\right)\left[\frac{1}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\beta_i-\beta\right)^2\bar{X}_{1iT}\right] \\
&\quad+\max_i\left(\frac{1}{\hat{\sigma}^2_{u,i}}\right)\left[\frac{1}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\beta_i-\beta\right)^2 E\left(X_{1iT}\right)\right] \\
&= O_p\left(T^{\varepsilon_2}\right)O_p\left(\frac{1}{\sqrt{n}}\right)+O_p\left(T^{\varepsilon_2}\right)O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}
$$

Thus, $C_2+C_8=O_p\left(T^{\varepsilon_2}/\min\left\{\sqrt{n},\sqrt{T}\right\}\right)$; by definition, this term is present only if $\lambda < 1$. Let us now turn our attention to $C_3 + C_9$; we have

$$\begin{aligned}
C_3 + C_9 &= \frac{\left(\hat{\beta} - \beta\right)^2}{n} \sum_{i=1}^{n} \left(\frac{1}{\hat{\sigma}_{u,i}}\right)^2 X_{1iT} \\
&= \frac{\left(\hat{\beta} - \beta\right)^2}{n} \sum_{i=1}^{\lfloor n\lambda \rfloor} \left(\frac{1}{\hat{\sigma}_{u,i}}\right)^2 X_{1iT} + \frac{\left(\hat{\beta} - \beta\right)^2}{n} \sum_{i=\lfloor n\lambda \rfloor+1}^{n} \left(\frac{1}{\hat{\sigma}_{u,i}}\right)^2 X_{1iT} \\
&= C_{39,1} + C_{39,2}.
\end{aligned}$$

Recalling that $\hat{\beta} - \beta = O_p\left(n^{-1/2}\right)$, similar arguments as above lead to $C_{39,1} = O_p\left(n^{-3/2}\right) + O_p\left(n^{-1}T^{-1/2}\right)$, and $C_{39,2} = O_p\left(T^{\varepsilon_2} / \min\left\{n\sqrt{n}, n\sqrt{T}\right\}\right)$. Consider $C_4$, and define $X_{2iT} = \left(\beta_i - \beta\right)\left(T^{-4}\sum_{t=1}^{T} \bar{x}_{it}^2\right)^2$

$$\begin{aligned}
-\sigma_x^2 \frac{C_4}{180} &= \frac{\left(\hat{\beta} - \beta\right)}{n} \sum_{i=1}^{\lfloor n\lambda \rfloor} \left(\frac{1}{\hat{\sigma}_{u,i}} X_{2iT}\right) + \frac{\left(\hat{\beta} - \beta\right)}{n} \sum_{i=\lfloor n\lambda \rfloor+1}^{n} \left(\frac{1}{\hat{\sigma}_{u,i}} X_{2iT}\right) \\
&= C_{4,1} + C_{4,2}.
\end{aligned}$$

Since $E\left(X_{2iT}\right) = E\left(\beta_i - \beta\right) E\left[\left(T^{-4}\sum_{t=1}^{T} \bar{x}_{it}^2\right)^2\right] = 0$ for all $i$ and $T$, the MDS CLT ensures that $\sum_{i=1}^{\lfloor n\lambda \rfloor}\left(X_{2iT}/\hat{\sigma}_{u,i}\right) = O_p\left(\sqrt{n}\right)$, and thus $C_{4,1} = O_p\left(n^{-1}\right)$. Also

$$\begin{aligned}
C_{4,2} &\leq \frac{\left(\hat{\beta} - \beta\right)}{n} \max_i \left|\frac{1}{\hat{\sigma}_{u,i}}\right| \sum_{i=\lfloor n\lambda \rfloor+1}^{n} X_{2iT} \\
&= \frac{1}{n} O_p\left(\frac{1}{\sqrt{n}}\right) O_p\left(T^{\varepsilon_2}\right) O_p\left(\sqrt{n}\right) = O_p\left(\frac{T^{\varepsilon_2}}{n}\right).
\end{aligned}$$

Thus, $C_4 = O_p\left(n^{-1}T^{\varepsilon_2}\right)$. Turning our attention to $C_5$, and defining $X_{3iT} = \left(\beta_i - \beta\right)\left(T^{-2}\sum_{t=1}^{T} \bar{x}_{it}\bar{u}_{it}^{+}\right)\left(T^{-2}\sum_{t=1}^{T} \bar{x}_{it}^2\right)$, we have

$$\begin{aligned}
\sigma_x^2 \frac{C_5}{180} &= \frac{1}{n} \sum_{i=1}^{\lfloor n\lambda \rfloor} \left(\frac{1}{\hat{\sigma}_{u,i}} X_{3iT}\right) + \frac{1}{n} \sum_{i=\lfloor n\lambda \rfloor+1}^{n} \left(\frac{1}{\hat{\sigma}_{u,i}} X_{3iT}\right) \\
&= C_{5,1} + C_{5,2}.
\end{aligned}$$

Since $E\left(X_{3iT}\right)=0$ for all $i$ and $T$, the MDS CLT yields $C_{5,1}=O_p\left(n^{-1/2}\right)$. When $\bar{u}_{it}$ is stationary, we have $X_{3iT}=O_p\left(T^{-1}\right)$, which implies

$$C_{5,2}\leq\max_i\left|\frac{1}{\hat{\sigma}_{u,i}}\right|\frac{1}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}X_{3iT}=O_p\left(T^{\varepsilon_2}\right)O_p\left(\frac{1}{T}\right)O_p\left(\frac{1}{\sqrt{n}}\right).$$

Therefore, $C_5=O_p\left(n^{-1/2}\right)+O_p\left(n^{-1/2}T^{\varepsilon_2-1}\right)$. As far as $C_6$ is concerned, let $X_{4iT}=\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)$

$$\begin{aligned}\sigma_x^2\frac{C_6}{180} &= \frac{\left(\hat{\beta}-\beta\right)}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{1}{\hat{\sigma}_{u,i}}X_{4iT}\right)+\frac{\left(\hat{\beta}-\beta\right)}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\frac{1}{\hat{\sigma}_{u,i}}X_{4iT}\right)\\ &= C_{6,1}+C_{6,2}.\end{aligned}$$

Note that $E\left(X_{4iT}\right)=O\left(T^{-1/2}\right)$ when $\bar{u}_{it}$ is nonstationary and $E\left(X_{4iT}\right)=O\left(T^{-3/2}\right)$ when $\bar{u}_{it}$ is stationary. Then we have

$$\begin{aligned}C_{6,1} &= \frac{\left(\hat{\beta}-\beta\right)}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{1}{\hat{\sigma}_{u,i}}\bar{X}_{4iT}\right)+\frac{\left(\hat{\beta}-\beta\right)}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left[\frac{1}{\hat{\sigma}_{u,i}}E\left(X_{4iT}\right)\right]\\ &= C_{6,1,1}+C_{6,1,2},\end{aligned}$$

where $\bar{X}_{4iT}=X_{4iT}-E\left(X_{4iT}\right)$. Then, recalling that $\hat{\beta}-\beta=O_p\left(n^{-1/2}\right)$, applying the MDS CLT yields $C_{6,1,1}=O_p\left(n^{-1}\right)$ and

$$\begin{aligned}C_{6,1,2} &\leq \max_i\left|E\left(X_{4iT}\right)\right|\frac{\left(\hat{\beta}-\beta\right)}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\frac{1}{\hat{\sigma}_{u,i}}\\ &= O\left(\frac{1}{\sqrt{T}}\right)O_p\left(\frac{1}{\sqrt{n}}\right);\end{aligned}$$

hence, $C_{6,1,2}=O_p\left(n^{-1/2}T^{-1/2}\right)$, and thus $C_{6,1}=O_p\left(n^{-1}\right)+O_p\left(n^{-1/2}T^{-1/2}\right)$. As far as $C_{6,2}$ is concerned, similar arguments as for $C_5$ entail $C_{6,2}=O_p\left(n^{-1/2}T^{\varepsilon_2-1}\right)+$

$O_p\left(T^{\varepsilon_2-3/2}\right)$. Consider now $C_7$

$$\begin{aligned}
C_7 &= \frac{12}{nT^2}\sum_{i=1}^{\lfloor n\lambda\rfloor}\sum_{t=1}^{T}\bar{u}_{it}^{+2}+\frac{12}{nT^2}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\sum_{t=1}^{T}\left(\frac{\bar{u}_{it}}{\hat{\sigma}_{u,i}}\right)^2 \\
&= C_{7,1}+C_{7,2}.
\end{aligned}$$

Similar arguments as for $C_{1,1}$ yield $C_{7,1}\xrightarrow{p}2\lambda+O_p\left(T^{-\varepsilon_1}\right)+O_p\left(T^{-1/2}\right)$. As far as $C_{10}$, $C_{11}$ and $C_{12}$ are concerned, these are similar to (respectively) $C_5$, $C_6$ and $C_4$, and therefore similar passages as above would prove that they have the same asymptotic magnitude.

Putting all together, we have

$$\hat{\lambda}=-\lambda+2\lambda+(1-d)\,O_p\left(\frac{1}{T^{\varepsilon_1}}\right)+(1-d_1)\left[O_p\left(\frac{T^{\varepsilon_2}}{\sqrt{T}}\right)+O_p\left(\frac{T^{\varepsilon_2}}{\sqrt{n}}\right)\right]+o_p\left(1\right),$$

which proves the theorem. ■

**Proof of Corollary 2.** Consider first the case $\lambda>0$, which corresponds to (18). As (34) and the passages thereafter show, as $(n,T)\to\infty$ with $\frac{n}{T}\to 0$, the terms that dominate are $A_5$ and $A_8$; thus, the limiting distribution of $\sqrt{n}\left(\hat{\sigma}_\beta^2-\sigma_\beta^2\right)$ is given by

$$\begin{aligned}
&\sqrt{n}\frac{30}{\sigma_x^2 nT^4}\sum_{i=1}^{n}\left[\left(\beta_i-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right]-\frac{2}{nT^2}\sum_{i=1}^{n}\left[\left(\beta_i-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right] \\
=\;&\sqrt{n}\frac{2}{n}\sum_{i=1}^{n}\left\{\left(\beta_i-\beta\right)\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\left[\frac{15}{\sigma_x^2T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)-1\right]\right\} \\
=\;&\frac{2}{\sqrt{n}}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left\{\left(\beta_i-\beta\right)\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\left[\frac{15}{\sigma_x^2T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)-1\right]\right\} \\
&+\frac{2}{\sqrt{n}}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left\{\left(\beta_i-\beta\right)\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\left[\frac{15}{\sigma_x^2T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)-1\right]\right\} \\
=\;&\sqrt{\lambda}\frac{2}{\sqrt{n\lambda}}\sum_{i=1}^{\lfloor n\lambda\rfloor}Y_{iT}+O_p\left(\frac{1}{T}\right).
\end{aligned}$$

The sequence $\{Y_{iT}\}_{i=1}^{\lfloor n\lambda \rfloor}$ is i.i.d. across $i$, and, for all $T$, it holds that

$$E\left(Y_{iT}\right)=E\left(\beta_i-\beta\right)\times E\left\{\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\left[\frac{15}{\sigma_x^2 T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)-1\right]\right\}=0.$$

Also, note that

$$E\left|Y_{iT}\right|^{2+\delta}=\left(\beta_i-\beta\right)^{2+\delta}\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^{2+\delta}\left[\frac{15}{\sigma_x^2 T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)-1\right]^{2+\delta},$$

which is finite in light of the proof of Lemma 2. Thus, as $(n,T)\to\infty$, an MDS CLT holds for $n^{-1/2}\sum_{i=1}^{n}Y_{iT}$ such that

$$\frac{1}{\sqrt{n\lambda}}\sum_{i=1}^{\lfloor n\lambda \rfloor}Y_{iT}\xrightarrow{d}\left[E\left(\lim_{T\to\infty}Y_{iT}^2\right)\right]^{1/2}\times Z=\sqrt{V_\sigma}\times Z,$$

with $Z\sim N\left(0,1\right)$ independent of $V_\sigma$. As far as $V_\sigma$ is concerned, as $T\to\infty$

$$Y_{iT}^2\overset{a.s.}{=}\left(\beta_i-\beta\right)^2\left(\sigma_u\sigma_x\int\bar{W}_1\bar{W}_2\right)^2\left[\left(15\int\bar{W}_1^2-1\right)^2\right]^2+o_p\left(1\right),$$

and thus

$$\lambda E\left(\lim_{T\to\infty}Y_{iT}^2\right)=\lambda\sigma_u^2\sigma_x^2\sigma_\beta^2\times E\left[\left(\int\bar{W}_1\bar{W}_2\right)^2\left(15\int\bar{W}_1^2-1\right)^2\right].$$

Thus,

$$\sqrt{n}\left(\hat{\sigma}_\beta^2-\sigma_\beta^2\right)\xrightarrow{d}2\sqrt{\lambda}\sqrt{\sigma_u^2\sigma_x^2\sigma_\beta^2}\times\sqrt{\delta_\sigma}\times Z.$$

As far as (20) is concerned, when $\lambda=0$, (34) and the passages thereafter entail that $\hat{\sigma}_\beta^2-\sigma_\beta^2=O_p\left(n^{-1}\right)$ as $(n,T)\to\infty$ with $\frac{n}{\sqrt{T}}\to 0$; the terms that dominate are, in this case, $A_4$ and $A_6$. Thus, the asymptotics of $n\left(\hat{\sigma}_\beta^2-\sigma_\beta^2\right)$

is driven by

$$\frac{1}{nT^2}\left(\hat{\beta}-\beta\right)^2\sum_{i=1}^{n}\left[\frac{15}{\sigma_x^2T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2-\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\right]$$
$$-\frac{30}{\sigma_x^2nT^4}\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\left[\left(\beta_i-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\right].$$

Letting $Y_{2iT}=T^{-2}\left(\beta_i-\beta\right)\sum_{t=1}^{T}\bar{x}_{it}^2$ with $E\left(Y_{2iT}\right)=0$, and recalling that

$$\frac{1}{n}\sum_{i=1}^{n}\left[\frac{15}{\sigma_x^2T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2-\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\right]\overset{a.s.}{=}\frac{1}{6}\sigma_x^2+o_p\left(1\right)$$

and that when $\lambda=0$ it holds that

$$\hat{\beta}-\beta\overset{a.s.}{=}\frac{6}{\sigma_x^2}\frac{1}{n}\sum_{i=1}^{n}Y_{2iT}+o_p\left(1\right),$$

we have that

$$\begin{aligned}
&n\left(\hat{\sigma}_\beta^2-\sigma_\beta^2\right)\\
&\overset{a.s.}{=}\frac{36}{\sigma_x^4}\left(\frac{1}{6}\sigma_x^2\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Y_{2iT}\right)^2-\frac{30}{\sigma_x^2}\frac{6}{\sigma_x^2}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Y_{2iT}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Y_{2iT}d_{2iT}\right)+o_p\left(1\right)\\
&\overset{a.s.}{=}\frac{6}{\sigma_x^2}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Y_{2iT}\right)\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Y_{2iT}\left(1-\frac{30}{\sigma_x^2}d_{2iT}\right)\right]+o_p\left(1\right),
\end{aligned}$$

with $d_{2iT}=T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2$. Note that $E\left(Y_{2iT}d_{2iT}\right)=0$. Also, $E\left|Y_{2iT}\right|^{2+\delta}<\infty$, and $E\left|Y_{2iT}d_{2iT}\right|^{2+\delta}=E\left|Y_{2iT}\right|^{2+\delta}E\left|d_{2iT}\right|^{2+\delta}<\infty$. Thus, as $(n,T)\to\infty$, the MDS CLT yields

$$n\left(\hat{\sigma}_\beta^2-\sigma_\beta^2\right)\overset{d}{\to}\frac{6}{\sigma_x^2}\sqrt{E\left(\lim_{T\to\infty}Y_{2iT}^2\right)}\sqrt{E\left[\lim_{T\to\infty}\left(Y_{2iT}^2\right)\left(1-\frac{30}{\sigma_x^2}d_{2iT}\right)^2\right]}\times Z^2,$$

with $Z \sim N(0,1)$; we have

$$E\left(\lim_{T\to\infty} Y_{2iT}^2\right) = E\left[(\beta_i - \beta)^2\right] E\left[\sigma_x^4 \left(\int \bar{W}^2\right)^2\right] = \frac{\sigma_\beta^2 \sigma_x^4}{45},$$

and

$$\begin{aligned} E\left[\lim_{T\to\infty}\left(Y_{2iT}^2\right)\left(1 - \frac{30}{\sigma_x^2} d_{2iT}\right)^2\right] &= E\left[(\beta_i - \beta)^2\right] E\left\{\left[\sigma_x^2 \int \bar{W}^2\right]^2 \left[1 - 30\int \bar{W}^2\right]^2\right\} \\ &= \sigma_\beta^2 \sigma_x^4 E\left[\left(\sigma_x^2 \int \bar{W}^2\right)^2 \left(1 - 30\int \bar{W}^2\right)^2\right]. \end{aligned}$$

Putting it all together, we have

$$n\left(\hat{\sigma}_\beta^2 - \sigma_\beta^2\right) \xrightarrow{d} \frac{6}{\sqrt{45}} \sigma_\beta^2 \sigma_x^2 \sqrt{E\left[\left(\sigma_x^2 \int \bar{W}^2\right)^2 \left(1 - 30\int \bar{W}^2\right)^2\right]} \times \chi_{(1)}^2.$$

■

**Proof of Theorem ??.** The results in the theorem follow immediately from (37). Under the null that $\lambda = 0$, it holds that

$$\widehat{\lambda\sigma_u^2} = O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{n}}\right) + o_p(1),$$

and thus $\widehat{\lambda\sigma_u^2} = O_p\left(1/\min\left\{\sqrt{n}, \sqrt{T}\right\}\right)$. Consider first the case whereby $\frac{n}{T} \to 0$, which entails $\widehat{\lambda\sigma_u^2} = O_p\left(n^{-1/2}\right)$. From (36), and from the passages thereafter, it follows that the term that dominates in the decomposition is $B_3$, given by

$$B_3 = \frac{1}{nT^2} \sum_{i=1}^n (\beta_i - \beta)^2 \left[2\left(\sum_{t=1}^T \bar{x}_{it}^2\right) - \frac{15}{\sigma_x^2 T^2}\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2\right].$$

Let $Z_{iT} = 2T^{-2}\left(\sum_{t=1}^T \bar{x}_{it}^2\right) - 15\left(\sigma_x^2 T^4\right)^{-1}\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2$. From the proof of

Lemma 2 we have, as $T \to \infty$

$$Z_{iT} \overset{a.s.}{=} 2\sigma_x^2 \int \bar{W}^2 - 15\sigma_x^2 \left( \int \bar{W}^2 \right)^2 + O_p\left( \frac{1}{\sqrt{T}} \right),$$

where $\bar{W}$ is a demeaned standard Brownian motion; (26) and (27) entail that, uniformly in $i$

$$E\left(Z_{iT}\right) = 2\sigma_x^2 \times \frac{1}{6} - 15\sigma_x^2 \times \frac{1}{45} + O\left( \frac{1}{\sqrt{T}} \right) = O\left( \frac{1}{\sqrt{T}} \right).$$

Thus, letting $\bar{Z}_{iT} = Z_{iT} - E\left(Z_{iT}\right)$, $B_3$ in (36) can be rewritten as

$$B_3 = \frac{1}{n}\sum_{i=1}^{n} \left(\beta_i - \beta\right)^2 \bar{Z}_{iT} + \frac{1}{n}\sum_{i=1}^{n} \left(\beta_i - \beta\right)^2 E\left(Z_{iT}\right) = B_{31} + B_{32}.$$

Consider $B_{31}$. Conditioning upon $C$, $\left(\beta_i - \beta\right)^2 \bar{Z}_{iT}$ is an iid sequence with $E\left[\left(\beta_i - \beta\right)^2 \bar{Z}_{iT}\right] = E\left[\left(\beta_i - \beta\right)^2\right] E\left(\bar{Z}_{iT}\right) = 0$. Also,

$$E\left|\left(\beta_i - \beta\right)^2 \bar{Z}_{iT}\right|^{2+\delta} = E\left|\left(\beta_i - \beta\right)^2\right|^{2+\delta} E\left|\bar{Z}_{iT}\right|^{2+\delta}.$$

Assumption 3 ensures that $E\left|\left(\beta_i - \beta\right)^2\right|^{2+\delta} < \infty$. As far as $E\left|\bar{Z}_{iT}\right|^{2+\delta}$ is concerned, note that

$$E\left|\bar{Z}_{iT}\right|^{2+\delta} \leq M_1 \left[ 2^{2+\delta} \left( \frac{1}{T^2}\sum_{t=1}^{T} \bar{x}_{it}^2 \right)^{2+\delta} + \left( \frac{15}{\sigma_x^2} \right)^{2+\delta} \left( \frac{1}{T^4}\sum_{t=1}^{T} \bar{x}_{it}^4 \right)^{2+\delta} \right],$$

and therefore in light of Assumption 2 and the passages in the proof of Lemma , $E\left|\bar{Z}_{iT}\right|^{2+\delta} < \infty$. Thus, $\left(\beta_i - \beta\right)^2 \bar{Z}_{iT}$ is an MDS that satisfies a Liapunov condition, and therefore the MDS CLT yields that $\sum_{i=1}^{n} \left(\beta_i - \beta\right)^2 \bar{Z}_{iT} = O_p\left(\sqrt{n}\right)$ with, as $(n, T) \to \theta$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \left(\beta_i - \beta\right)^2 \bar{Z}_{iT} \xrightarrow{d} \sqrt{E\left[\left(\beta_i - \beta\right)^4 \lim_{T\to\infty} \bar{Z}_{iT}^2\right]} \times Z,$$

with $Z \sim N(0,1)$ independent of $\bar{Z}_{iT}$. As $(n,T) \to \infty$, we have

$$
\begin{aligned}
E\left[(\beta_i - \beta)^4 \lim_{T\to\infty} \bar{Z}_{iT}^2\right] &= E\left[(\beta_i - \beta)^4\right] E\left[\lim_{T\to\infty} \bar{Z}_{iT}^2\right] \\
&= \kappa_\beta \times \sigma_x^4 E\left[\left(\int \bar{W}^2\right)^2 \left(2 - 15\int \bar{W}^2\right)^2\right].
\end{aligned}
$$

As far as $B_{32}$ is concerned, it holds that

$$
\begin{aligned}
B_{32} &\leq \max_i |E(Z_{iT})| \left[\frac{1}{n}\sum_{i=1}^n (\beta_i - \beta)^2\right] \\
&= O_p\left(\frac{1}{\sqrt{T}}\right),
\end{aligned}
$$

and thus it is dominated by $B_{31}$ under $\frac{n}{T} \to 0$. Finally, recalling that

$$
\frac{\widehat{\lambda\sigma_u^2}}{\sigma_x^2} = \widehat{\lambda\sigma_u^2} \times \frac{nT^2}{6\sum_{i=1}^n \sum_{t=1}^T \bar{x}_{it}^2},
$$

it holds that, under $H_0$ as $(n,T) \to \infty$ with $\frac{n}{T} \to 0$

$$
\sqrt{n} \times \frac{\widehat{\lambda\sigma_u^2}}{\sigma_x^2} \xrightarrow{d} \frac{1}{\sigma_x^2} \times \sqrt{\kappa_\beta \sigma_x^4 \delta_\lambda} \times Z.
$$

When $\frac{T}{n} \to 0$, (37) entails $\sqrt{T} \times \frac{\widehat{\lambda\sigma_u^2}}{\sigma_x^2} = O_p(1)$. Last, consider $H_A^{(n,T)}$; since

$$
\min\left\{\sqrt{n}, \sqrt{T}\right\} \times \frac{\widehat{\lambda\sigma_u^2}}{\sigma_x^2} \overset{a.s.}{=} \min\left\{\sqrt{n}, \sqrt{T}\right\} \times \frac{\lambda\sigma_u^2}{\sigma_x^2} + D_\lambda,
$$

the drift term is nonzero as $(n,T) \to \infty$ if

$$
\frac{\lambda\sigma_u^2}{\sigma_x^2} = \frac{1}{\min\left\{\sqrt{n}, \sqrt{T}\right\}}.
$$

■

**Proof of Theorem ??.** The main result for the proof is (34), which will be constantly referred to henceforth. From (35), it emerges that

$$\hat{\sigma}_\beta^2 - \sigma_\beta^2 = O_p\left(\frac{1}{\sqrt{T}}\right) + d_\sigma O_p\left(\frac{1}{\sqrt{n}T^d}\right) + O_p\left(\frac{1}{n}\right) + o_p(1);$$

thus, for $\lambda(0,1]$ and $\sigma_\beta^2 = 0$, the term that dominates when $(n,T) \to \infty$ (under $\frac{n}{\sqrt{T}} \to 0$) is of order $O_p(n^{-1})$. As (34) and the passages thereafter show, the terms that dominate are $A_4$, $A_7$ and $A_9$; thus, the limiting distribution of $n\hat{\sigma}_\beta^2$ is given by

$$\frac{1}{n}\left(\hat{\beta}-\beta\right)^2 \sum_{i=1}^n \left[\frac{15}{\sigma_x^2 T^4}\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2 - \left(\frac{1}{T^2}\sum_{t=1}^T \bar{x}_{it}^2\right)\right]$$
$$+\frac{1}{n}\left(\hat{\beta}-\beta\right)\sum_{i=1}^n \left[\left(\frac{1}{T^2}\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)\left(2 - \frac{30}{\sigma_x^2 T^2}\sum_{t=1}^T \bar{x}_{it}^2\right)\right].$$

Let $Y_{iT} = T^{-2}\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}$, and note that in light of Lemma 2 it holds

$$\frac{1}{n}\sum_{i=1}^n \left[\frac{15}{\sigma_x^2 T^4}\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2 - \left(\frac{1}{T^2}\sum_{t=1}^T \bar{x}_{it}^2\right)\right] \overset{a.s.}{=} \frac{1}{6}\sigma_x^2 + o_p(1);$$

also,

$$\hat{\beta} - \beta \overset{a.s.}{=} \frac{6}{\sigma_x^2}\left[\frac{1}{nT^2}\sum_{i=1}^n Y_{iT}\right] + o_p(1).$$

Thus, after some algebra, the limiting distribution of $n\left(\hat{\sigma}_\beta^2 - \sigma_\beta^2\right)$ is driven by

$$n\lambda \times \frac{6}{n\sigma_x^2}\left[\frac{1}{\sqrt{n\lambda}}\sum_{i=1}^{\lfloor n\lambda \rfloor} Y_{iT}\right]\left[\frac{1}{\sqrt{n\lambda}}\sum_{i=1}^{\lfloor n\lambda \rfloor} Y_{iT}(1+d_{iT})\right] + o_p(1),$$

with

$$d_{iT} = 2 - \frac{30}{\sigma_x^2 T^2}\sum_{t=1}^T \bar{x}_{it}^2.$$

Thus, the MDS CLT yields

$$n\hat{\sigma}_{\beta}^{2} \xrightarrow{d} \frac{6\lambda}{\sigma_x^2}\left[E\left(Y_{iT}^2\right)\right]^{1/2}\left\{E\left[Y_{iT}^2\left(1+d_{iT}\right)^2\right]\right\}^{1/2}\times Z^2,$$

where $Z \sim N(0,1)$. As $T\to\infty$

$$E\left(Y_{iT}^2\right) = E\left[\lim_{T\to\infty}\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2\right] = \frac{\sigma_u^2\sigma_x^2}{90},$$

and

$$\begin{aligned} E\left[Y_{iT}^2\left(1+d_i\right)^2\right] &= E\left[\lim_{T\to\infty}\left(3-\frac{30}{\sigma_x^2 T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2\right] \\ &= \sigma_u^2\sigma_x^2 E\left[\left(\int \bar{W}_1\bar{W}_2\right)^2\left(3-30\left(\int \bar{W}_1^2\right)\right)^2\right], \end{aligned}$$

and thus

$$\begin{aligned} n\hat{\sigma}_{\beta}^{2} &\overset{a.s.}{=} \frac{6\lambda}{\sigma_x^2}\sigma_u^2\sigma_x^2\sqrt{\frac{1}{90}}\times\sqrt{9E\left[\left(\int \bar{W}_1\bar{W}_2\right)^2\left(1-10\left(\int \bar{W}_1^2\right)\right)^2\right]}\times Z^2+o_p(1) \\ &= \frac{6}{\sqrt{10}}\lambda\sigma_u^2\sqrt{E\left[\left(\int \bar{W}_1\bar{W}_2\right)^2\left(1-10\left(\int \bar{W}_1^2\right)\right)^2\right]}\times Z^2+o_p(1). \end{aligned}$$

As $(n,T)\to\infty$ with $\frac{\sqrt{T}}{n}\to 0$, (35) leads to $\sqrt{T}\left(\hat{\sigma}_{\beta}^2-\sigma_{\beta}^2\right)=O_p(1)$. Last, consider $H_A^{(n,T)}$; since

$$\min\left\{n,\sqrt{T}\right\}\times\hat{\sigma}_{\beta}^2 \overset{a.s.}{=} \min\left\{n,\sqrt{T}\right\}\times\sigma_{\beta}^2+D_{\sigma},$$

the drift term is nonzero as $(n,T)\to\infty$ if

$$\sigma_{\beta}^2=\frac{1}{\min\left\{n,\sqrt{T}\right\}}.$$

■